APPLICATIONS OF QUANTUM GROUPS TO CONFORMALLY INVARIANT RANDOM GEOMETRY

Eveliina Peltola

Academic dissertation

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Sincerely,
Eveliina Peltola
Cambridge, May 2016

In this thesis

This thesis includes an introductory part and the following articles.

[A] K. Kytölä and E. Peltola. Conformally covariant boundary correlation functions with a quantum group. Preprint in `arXiv:1408.1384`, 2014.

[B] K. Kytölä and E. Peltola. Pure partition functions of multiple SLEs. To appear in *Communications in Mathematical Physics*†. Preprint in `arXiv:1506.02476`, 2015.

[C] A. Karrila, K. Kytölä, and E. Peltola. Conformal blocks, pure partition functions, and Kenyon-Wilson binary relation. Manuscript.

[D] E. Peltola. Basis for solutions of the Benoit & Saint-Aubin PDEs with particular asymptotic properties. Preprint in `arXiv:1605.06053`, 2016.

† The final publication will be available at Springer via `http://dx.doi.org/10.1007/s00220-016-2655-2`.

Introduction

Contents

1. Overview

This thesis concerns questions motivated by two-dimensional critical lattice models of statistical mechanics, and conformal field theory (CFT). The core idea is to apply algebraic techniques to solve questions in random geometry, and reveal the algebraic structures therein. In this thesis, we consider the interplay between braided Hopf algebras (quantum groups) and CFT, with applications to critical lattice models and conformally invariant random curves, Schramm-Loewner evolutions (SLE).

In the first article [A], the general question is to construct explicit expressions for CFT correlation functions — analytic functions of several complex variables satisfying linear homogeneous partial differential equations known as the Benoit & Saint-Aubin PDEs. Such PDEs emerge in CFT from singular vectors in representations of the Virasoro algebra, the conformal symmetry algebra. The correlation functions of CFT are believed to describe scaling limits of correlations in critical lattice models, expected to exhibit conformal invariance in the scaling limit. Rigorous results towards the conformal invariance of the scaling limits of a number of lattice models have now been established, but many open questions remain.

The random curves known as SLEs were Oded Schramm's groundbreaking idea in 1999: his proposal for scaling limits of interfaces of critical lattice models. Remarkably, special cases of the Benoit & Saint-Aubin PDEs also arise in the theory of SLEs, from vanishing drift terms of local martingales with respect to the random SLE curve. This is no coincidence: it reflects the evidence that SLE and CFT have a deep connection, which has not yet, however, been fully revealed.

The article [B] contains applications of the results of [A] to questions in the theory of SLEs: the pure partition functions of multiple SLEs (processes of several interacting SLE curves), and the chordal SLE boundary visit probability amplitudes, also known as Green's functions. The relevant solutions to the PDEs are found by imposing certain natural boundary conditions given by specified asymptotic behavior. Loosely speaking, the appropriate boundary conditions can be deduced from the qualitative properties of the associated stochastic processes, or alternatively, by CFT fusion arguments. More general solutions to the PDEs are constructed in the article [D], in the spirit of fusion of CFT. The work in [D, E] pertains to the full characterization of the solution space of the PDEs, with a natural algebraic structure.

The above type of solutions emerge also from critical lattice models, as (conjectured) scaling limits of renormalized probabilities of crossing and boundary visit events of interfaces. In the article [C], we study such questions for the loop-erased random walk (LERW) and the uniform spanning tree (UST). We find explicit formulas for the probabilities and prove their convergence in the scaling limit to solutions of PDEs of Benoit & Saint-Aubin type. We also relate these functions to the conformal blocks of CFT.

To consider the asymptotics of the solutions, quantum groups play an important role. The article [A] is devoted to the development of a method based on the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. This hidden algebraic structure provides tools to directly read off properties of the solutions from representation theoretical data. This method is exploited in applications in subsequent work [B, D, E, G].

The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ can be regarded as a braided Hopf algebra. Therefore, its representation theory can also be applied when investigating monodromy properties of the solutions. In the articles [F, G], we consider questions related to the monodromy of the solutions, and algebraic structures arising thereof. Importantly, we reveal the structure of the monodromy representation of the solutions and prove the uniqueness of a monodromy invariant (i.e., single-valued) solution.

This introductory part is organized as follows. In the first part, we discuss statistical physics and critical phenomena, which serve as a motivation for the research of this thesis. We consider scaling limits of critical models and introduce Schramm-Loewner evolutions and notions of stochastic analysis related thereof. In the second part, we discuss the algebraic topics of this thesis: Hopf algebras, representation theory and some aspects of CFT and vertex operators. Finally, in Section 8, we give an overview of the contributions of this thesis to the understanding of random geometry and its algebraic content in interrelation with quantum groups, SLE, and CFT.

Part I: Scaling limits and Schramm-Loewner evolutions

An important motivation of the results of this thesis is the study of scaling limits of critical lattice models of statistical mechanics. We therefore begin this introductory part with examples of such models and explain how critical phenomena arise from their features. We then discuss scaling limits, that is, continuum limits of discrete lattice models. It has been predicted by physicists that the scaling limits are quantum field theories which at criticality enjoy a strong symmetry, conformal invariance. However, great difficulties arise when trying to rigorously consider the scaling limits, and mathematical results have only been established for but a few lattice models in two dimensions. In this thesis, we study random curves called Schramm-Loewner evolutions (SLE), which are scaling limits of critical interfaces, at least conjecturally (the convergence is proven in some cases). We discuss SLEs in Section 3.

2. Lattice models and critical phenomena

Lattice models are theories defined on discrete grids, used to model phenomena in condensed matter physics, chemistry, economics, and so on. They can also be regarded as discretizations of continuum models. Compared to continuum theories, lattice models are more tractable e.g. in terms of combinatorics, and many of them are exactly solvable. They share a common feature, randomness: the configurations of the system are random, and the physical variables are random variables on some probability space.

2.1. **Loop-erased random walk.** The symmetric random walk (SRW) has been used e.g. as an elementary model for a polymer. Geometrically, a SRW is a random path obtained by taking successive steps on the grid to neighboring vertices with equal probability. A trajectory of a SRW can be defined as $\omega_k = X_0 + X_1 + \ldots + X_k$ for $k \in \mathbb{N}$, started from $\omega_0 = X_0$, with random steps X_i that are independent and uniformly distributed on the set of all the possible directions of the step. More precisely, on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$, the SRW is a Markov process on the vertex set $\mathcal{V}$ whose transition probability from $v \in \mathcal{V}$ to $v' \in \mathcal{V}$ equals $\deg(v)^{-1}$ if $\langle v, v' \rangle \in \mathcal{E}$ is an edge of the graph $\mathcal{G}$, and zero otherwise. For instance, on the hypercubic grid with $\mathcal{V} = \mathbb{Z}^d$ and $\mathcal{E} = \big\{ \langle x, y \rangle \mid x, y \in \mathbb{Z}^d, \ |x - y| = 1 \big\}$, the transition probability from any vertex $x \in \mathbb{Z}^d$ to its neighbor $y \in \mathbb{Z}^d$ equals $(2d)^{-1} = \deg(x)^{-1}$.

Polymers usually should not cross themselves, but the symmetric random walk is self-crossing in small dimensions (on $\mathbb{Z}^d$ with $d \leq 4$), see e.g. [Dur10]. Therefore, a slightly modified polymer model, the self-avoiding walk (SAW) was introduced, by the chemist Paul Flory in the 1950s. It is defined as a SRW with the additional constraint that the path must not hit itself. While the symmetric random walk is well-understood, the self-avoiding walk is a rather difficult object to study, and even very elementary questions about it remain open — see e.g. [Law80, BDCGS11, Law16].

Because of the severe difficulties in understanding the behavior of the self-avoiding walk in two dimensions, Greg Lawler introduced the loop-erased random walk (LERW) as a toy model for the SAW [Law99]. However, it turned out that the SAW and LERW in fact belong to different universality classes — in dimensions $d \leq 4$, their behavior is different at macroscopic scales. Nevertheless, the LERW has now been extensively studied, and it is also closely related to other models, such as the uniform spanning tree (UST), domino tilings, and the Q-state Potts model as $Q \to 0$, see e.g. [Ken00, Gri09]. In this thesis, we consider the LERW and UST in the article [C], see also Section 8.5.

To construct a loop-erased walk γ, one takes a walk and erases all the loops along as they appear — see Figure 2.1. More precisely, if $\omega = (\omega_k)_{k=0}^{\ell}$ is a walk of length ℓ, its loop-erasure $\mathrm{LE}(\omega)$ is defined recursively as follows. Set $\mathrm{LE}(\omega)_0 := \omega_0$, denote by $k_0 := \max \{ j \leq \ell \mid \omega_j = \omega_0 \}$, and set $\mathrm{LE}(\omega)_1 := \omega_{k_0+1}$. Then, if $k_i < \ell$, define $k_{i+1} := \max \{ j \leq \ell \mid \omega_j = \omega_{k_i+1} \}$ and set $\mathrm{LE}(\omega)_{i+1} := \omega_{k_{i+1}}$. The process ends when $\mathrm{LE}(\omega)_j = \omega_\ell$ for some $j \leq \ell$. The constructed walk $\gamma = \mathrm{LE}(\omega)$ is self-avoiding.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a planar square grid, $\mathcal{G} \subsetneq \mathbb{Z}^2$, and denote by $\partial \mathcal{V} := \{ x \in \mathcal{V} \mid \mathrm{dist}(x, \mathbb{Z}^2 \setminus \mathcal{V}) = 1 \}$ the boundary vertices of $\mathcal{G}$ and by $\mathcal{V}^\circ := \mathcal{V} \setminus \partial \mathcal{V}$ the interior vertices. We consider the planar SRW on $\mathcal{G}$, started from a given vertex $\xi \in \mathcal{V}^\circ$, stopped at the first instant of hitting the boundary $\partial \mathcal{V}$, and

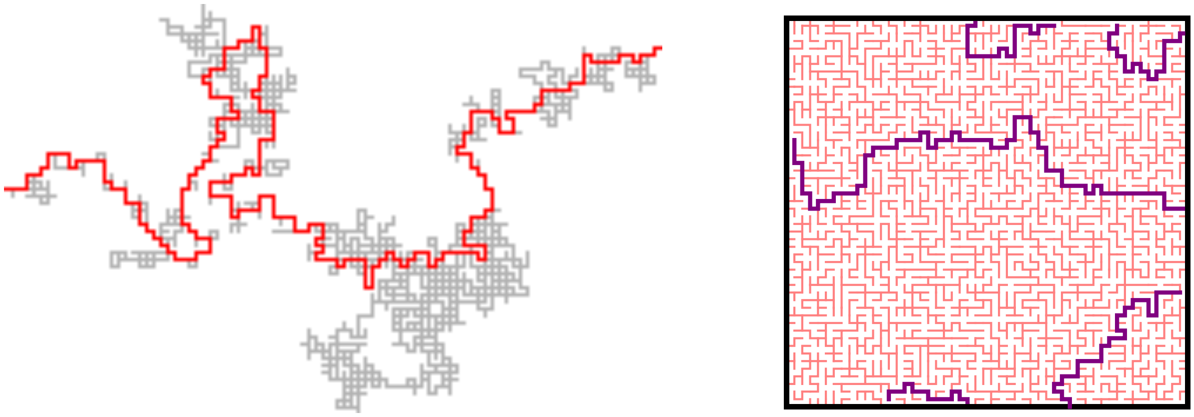


FIGURE 2.1. Simulations of the LERW and UST on a square grid. The left figure depicts a LERW (in red) obtained by taking a SRW (in grey) and erasing the loops. The right figure depicts a UST with wired boundary conditions (i.e., the boundary vertices are thought of as one vertex) where some boundary branches are highlighted. These boundary branches are distributed as mutually avoiding LERWs, in the sense of Wilson's algorithm [Wil96] and the connectivity probability (2.1).

conditioned to exit $\mathcal{G}$ through a given set $\mathcal{S} \subset \partial\mathcal{V}$. The planar loop-erased random walk (LERW) on $\mathcal{G}$ from ξ to $\mathcal{S}$ is defined as the loop-erasure $\gamma = \mathrm{LE}(\omega)$ of the symmetric random walk ω from ξ to $\mathcal{S}$.

2.2. **Uniform spanning tree and Fomin's formula.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite connected graph, with vertices $\mathcal{V}$ and edges $\mathcal{E}$; for instance, $\mathcal{V} = [-L, L]^2 \cap \mathbb{Z}^2$ and $\mathcal{E} = \{\langle x, y\rangle \mid x, y \in \mathcal{V},\ |x - y| = 1\}$. A spanning tree of $\mathcal{G}$ is a connected subgraph $\mathcal{T}$ with no cycles containing all the vertices of $\mathcal{G}$, see Figure 2.1. Since there are finitely many spanning trees on $\mathcal{G}$, we may choose one uniformly at random — the uniform distribution on spanning trees $\mathcal{T}$ of $\mathcal{G}$ is called the uniform spanning tree (UST) on $\mathcal{G}$.

A sample of the UST on $\mathcal{G}$ can be constructed from LERWs on $\mathcal{G}$ using Wilson's algorithm [Wil96] as follows (see also Figures 2.1 and 2.2). One first chooses two distinct vertices $v_0, v_1 \in \mathcal{V}$ and runs a LERW γ_0 between them. One then chooses another vertex v_2 which does not lie on γ_0 and runs a LERW γ_1 starting from v_2 until it hits γ_0. Continuing this, one obtains a spanning tree $\mathcal{T}$ on $\mathcal{G}$. David Wilson showed in [Wil96] that the choice of the vertices $v_0, v_1, v_2, \ldots$ does not matter and that the resulting spanning tree $\mathcal{T}$ is indeed uniformly distributed amongst all spanning trees on $\mathcal{G}$ — see also [Pem91].

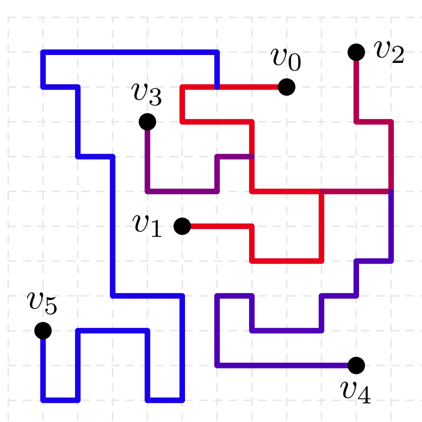


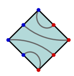
FIGURE 2.2. The first steps in Wilson's algorithm on a square grid.

Consider the UST with wired boundary conditions, that is, the boundary vertices $\partial\mathcal{V}$ are thought of as one single vertex v_∂. For any vertex $\xi \in \mathcal{V}^\circ$, there exists a unique branch of the UST connecting ξ to the boundary $\partial\mathcal{V}$. By Wilson's algorithm, the probability of the event that this branch reaches the boundary via a given edge $\overline{\eta} := \langle \eta, v_\partial \rangle$ is equal to the probability that a LERW on $\mathcal{G}$ from ξ to $\partial\mathcal{V}$ exits $\mathcal{G}$ through $\overline{\eta}$. Let then $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N \in \mathcal{V}$ be vertices next to the boundary, i.e., such that $\overline{\xi}_i$, $\overline{\eta}_i$ are boundary edges. Consider the probability of the event that, for all i, the boundary branch of the UST from ξ_i reaches the boundary via the edge $\overline{\eta}_i$. By applying Wilson's algoritm to grow the boundary branches one by one, see e.g. [C, Lemma 3.8], one sees that the probability of this connectivity event is

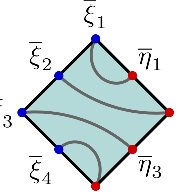
(2.1) $\mathbb{P}[\, \xi_i \text{ connects to } \overline{\eta}_i \text{ for all } i \,]$

$$= \sum_{\substack{\text{SRWs } \omega_1,\ldots,\omega_N \text{ on } \mathcal{G} \\ \text{from } \xi_1,\ldots,\xi_N \text{ to } \partial\mathcal{V}}} \mathbb{P}[\, \omega_i \text{ exits } \mathcal{G} \text{ through } \overline{\eta}_i \text{ for all } i, \text{ and } \mathrm{LE}(\omega_i) \cap \omega_j = \emptyset \text{ for all } i < j \,].$$

Suppose now that $\mathcal{G}$ is planar (e.g., $\mathcal{G} \subsetneq \mathbb{Z}^2$), and that the vertices $\xi_1, \ldots, \xi_N, \eta_N, \ldots, \eta_1 \in \mathcal{V}$ appear in counterclockwise order along the boundary (as illustrated below). Sergey Fomin [Fom01] observed that, in this situation, the connection probability (2.1) is in fact given by a determinant of the following type:

(2.2) $\mathbb{P}\Big[$ "  " $\Big] \;=\; \det\Big(\mathcal{K}(\xi_i, \eta_j)\Big)_{i,j=1}^{N}$, where

$$\mathcal{K}(\xi, \eta) \;=\; \sum_{\substack{\text{SRW } \omega \text{ on } \mathcal{G} \\ \text{from } \xi \text{ to } \partial\mathcal{V}}} \mathbb{P}[\, \omega \text{ exits } \mathcal{G} \text{ through } \overline{\eta} \,].$$



The kernel $\mathcal{K}(\xi, \eta)$ is called the discrete harmonic measure, also known as the discrete Poisson kernel. Generalizations and applications of Fomin's formula are treated the article [C] — see also Section 8.5.

2.3. **Gibbs measures.** In a general lattice model, the probability measure on the space Ω of all possible states of the system is called the Gibbs measure, or the Boltzmann distribution. Suppose that the space Ω of states is finite. The probability of a state $\sigma \in \Omega$ is given by the Boltzmann weight

$$\mathbb{P}[\sigma] \;:=\; \frac{1}{Z} e^{-\beta \mathcal{H}(\sigma)}, \tag{2.3}$$

where $\mathcal{H}$ is the Hamiltonian of the system, $\beta > 0$ a parameter (the inverse of the temperature $T = 1/\beta$), and the normalization constant Z, also called the partition function, is defined as $Z := \sum_{\sigma \in \Omega} e^{-\beta \mathcal{H}(\sigma)}$.

A prototypical example is the Ising model, in which the state space consists of spin configurations on a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, that is, assignments $\sigma \colon \mathcal{V} \to \{\pm 1\}$, with $\sigma_x = \pm 1$ attached to each vertex $x \in \mathcal{V}$. The probability of a given configuration σ is given by the Boltzmann weight (2.3) with the Hamiltonian

$$\mathcal{H}(\sigma) \;=\; -J \sum_{\langle x,y \rangle \in \mathcal{E}} \sigma_x \sigma_y \;-\; h \sum_{x \in \mathcal{V}} \sigma_x,$$

where the coupling parameter $J \in \mathbb{R}$ is the strength of the interaction of the spins and $h \in \mathbb{R}$ is the strength of the external magnetic field on the background of the model. The Hamiltonian describes the energy of the model.

The Ising model has been widely studied ever since Wilhelm Lenz and Ernst Ising, who introduced it in the 1920s. Recent breakthroughs in mathematics include the results on conformal invariance of the scaling limit of the critical two-dimensional Ising model, established using discrete complex analysis by Dima Chelkak, Stas Smirnov and collaborators [Smi10a, Hon10, Izy11, CS12, KS12, HS13, CI13, HK13, CDCH$^+$14, CHI15, KS15, Izy15, Izy16].

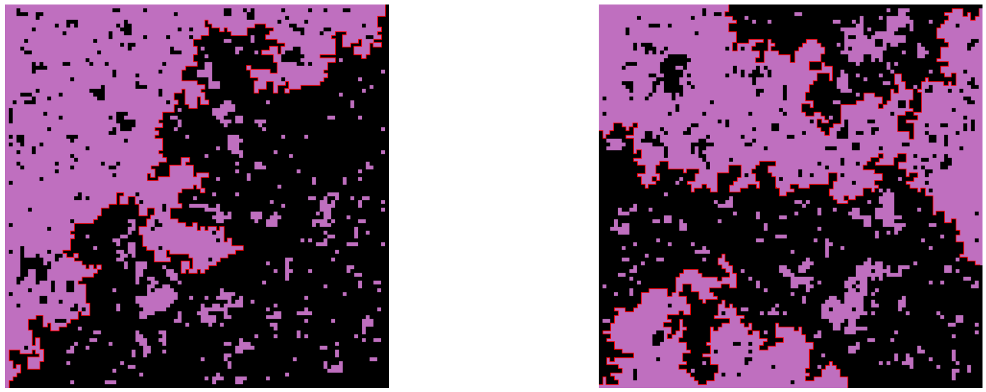


FIGURE 2.3. Simulations of the critical Ising model on a square grid with alternating boundary conditions. The black boundary arcs are fixed to the spin state +1 and the purple ones to −1. On the left, there are two complementary arcs and a single macroscopic interface emerges between them. On the right, the boundary conditions alternate at six boundary points and three interfaces emerge. Multiple interfaces have several possible topological configurations, depicted in Figure 2.5.

2.4. **Interfaces.** A very useful tool to study models such as the Ising model is to consider interfaces, that in the Ising model appear between spins which are not aligned. When $J > 0$, the Boltzmann distribution favors aligned spins and the appearance of an interface increases the energy of the system, by an additive constant proportional to the length of the interface. The system therefore tries to minimize the total length of the interfaces. However, one can force macroscopic random interfaces to appear by imposing boundary conditions. For example, if one conditions a segment of the boundary to have spins fixed to a particular state and the complementary segment to have the opposite spins, then a macroscopic interface emerges between the boundary points where the boundary conditions change — see Figure 2.3. Imposing more alternating segments, one obtains several interfaces, depicted in Figures 2.3 and 2.5.

Discrete interfaces have naturally the following domain Markov property (see also Figure 3.1). Consider an exploration process in $\mathcal{G}$, which e.g. for the Ising model is defined by following the interface between the opposite spins step by step, starting from a point on the boundary where the boundary conditions change, so that on the left one always has negative spins and on the right positive spins, say. One eventually hits the boundary at another point where the boundary conditions change. Let $\gamma(k)$, for $k = 0, 1, \ldots, n$, denote the exploration process in $\mathcal{G}$. Explore an interface up to some time k_0. Consider the exploration process $\tilde{\gamma}$ for the model on the smaller grid $\tilde{\mathcal{G}} = \mathcal{G} \setminus \gamma[0, k_0]$, started from the tip $\gamma(k_0)$, where the boundary conditions are taken as before on $\partial\mathcal{V}$ and naturally continued to both sides of the segment $\gamma[0, k_0]$ of $\partial\tilde{\mathcal{G}}$ (in our example, negative on the left of $\gamma[0, k_0]$ and positive on the right). The domain Markov property means that the distribution of the exploration process $\tilde{\gamma}$ associated to the model on $\tilde{\mathcal{G}}$ equals the conditional law of the original process γ in $\mathcal{G}$ given the initial segment $\gamma[0, k_0]$.

2.5. **Criticality.** The thermodynamic limit provides Gibbs measures on infinite state spaces. In this limit, the size of the system is taken to infinity while the lattice spacing remains fixed: e.g., for a system in a finite box $\mathcal{G}_L = [-L, L]^d \cap \mathbb{Z}^d$, one takes $L \to \infty$, so $\mathcal{G}_L \to \mathbb{Z}^d$. In infinite discrete systems, critical behavior may appear. For instance, Rudolf Peierls proved in 1936 that the Ising model (with $d \geq 2$ and $J > 0$) has a continuous phase transition: in low temperatures, the system exhibits spontaneous magnetization, but not in high temperatures — see Figure 2.4. The phase transition occurs at a unique critical temperature T_c, and the magnetization is a continuous function of the temperature T.

Another feature of criticality is the divergence of the correlation length at T_c in continuous phase transitions. The correlation between spins at x and y is the correlation[1] of the random spin variables σ_x

[1] The correlation of two random variables X and Y on the same probability space is defined as $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

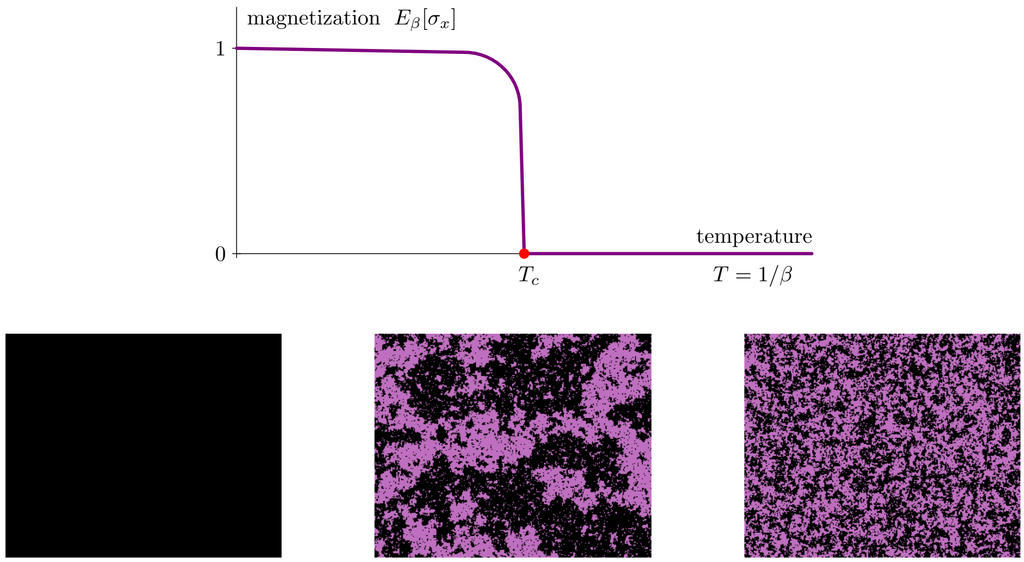


FIGURE 2.4. The phase transition of the ferromagnetic ($J > 0$) Ising model on $\mathbb{Z}^d$. The magnetization at x is defined as the expected value $\mathbb{E}_\beta[\sigma_x]$ of the spin σ_x at temperature $T = 1/\beta$. In high temperatures (right), the system is disordered, and both $\sigma_x = \pm 1$ are equally likely, whereas in low temperatures (left), aligned spins are favored, and the magnetization is non-zero. At criticality, macroscopic clusters of aligned spins appear. The magnetization is continuous across the phase transition at T_c.

and σ_y. In off-critical temperatures, the correlations decay exponentially fast in the distance of x and y: as $\sim e^{-\frac{1}{\xi}|x-y|}$, where $\xi = \xi(T)$ is the correlation length. When approaching the critical temperature, long-range fluctuations appear and the correlation length diverges: $\xi(T) \to \infty$ as $T \to T_c$. Then, the correlations have only a power law decay $\sim |x-y|^{-b}$ where $b > 0$ is a critical exponent of the model.

2.6. **Scaling limits and conformal invariance.** When the discrete model is thought of as an approximation of a continuum system, one is led to consider its scaling limit. Keeping the size of the system fixed, e.g., considering the model in a box $\mathcal{G}_\delta = [-L, L]^d \cap \delta\mathbb{Z}^d$ of size of order L^d, but letting the lattice spacing δ become smaller, one can approximate a continuum domain: $\mathcal{G}_\delta \to [-L, L]^d$ as $\delta \to 0$. Such a limit is called a scaling limit — one “zooms out” infinitely far to pass to a continuum model.

In physics, the scaling limit of a lattice model is described by a quantum field theory (QFT). The physical variables of the model are expected to converge to fields in the limiting QFT and the physical observables should converge to correlation functions of the fields, discussed more in Section 6. Also, interfaces of the discrete models should converge to continuum random curves, discussed more in Section 3.

Infinite discrete models are usually invariant under translations and rotations (by certain angles, e.g. the Ising model on a square grid has invariance under 90 degree rotations). Furthermore, in the scaling limit at criticality, the system exhibits self-similar behavior, in the sense that if one “zooms in”, the model is similar — the scaling limit is scale-invariant[2]. One also expects that the scaling limits of critical models with only short-range interactions enjoy invariance under local scalings and rotations. It is therefore reasonable to conjecture that their scaling limits enjoy full conformal invariance — conformal maps look locally like scalings and rotations (see Section 3.1 for more details). This was suggested by physicists already in the 1960s, see e.g. [Pol70, BPZ84a, BPZ84b, Car96].

[2]In the language of renormalization, critical models correspond to fixed points of the renormalization group flow: by self-similarity, the step by step averaging procedure of renormalization does not affect the critical model, see e.g. [Car96].

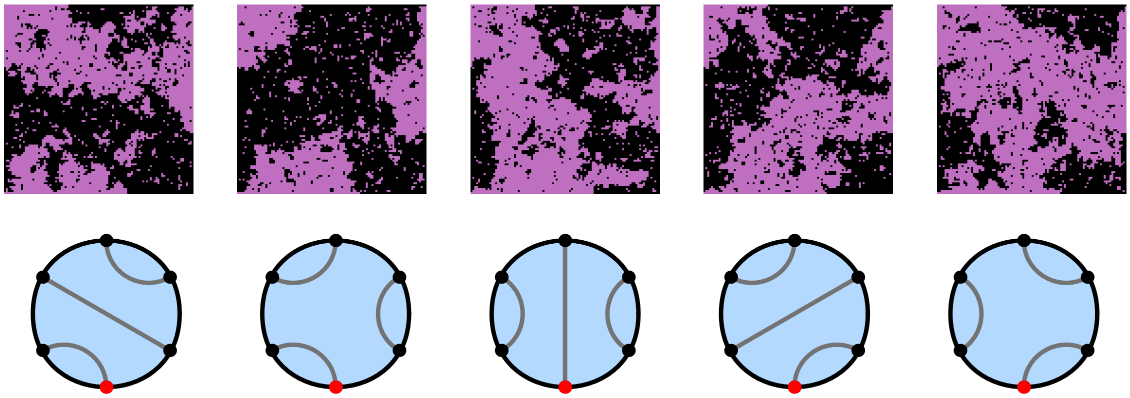


Figure 2.5. In lattice models with alternating boundary conditions on $2N$ boundary segments, interfaces can form several possible planar connectivities. The number of possibilities is the Catalan number $\mathrm{C}_N = \frac{1}{N+1}\binom{N}{2N}$. This figure shows the $\mathrm{C}_3 = 5$ connectivities of three interfaces in the critical Ising model on a square grid.

In the two-dimensional case, the powerful tools of complex analysis have provided success in the mathematical treatment of scaling limits of critical systems. Indeed, this has been a very active area of research in the last decades. In this thesis, we consider questions related to scaling limits of critical interfaces between boundary points, as well as certain boundary correlations in two-dimensional critical systems.

3. Schramm-Loewner evolution

Oded Schramm considered scaling limits of the planar loop-erased random walk and uniform spanning tree in his celebrated paper [Sch00]. He introduced the random curves now known as Schramm-Loewner evolutions (SLE) as conjectured scaling limits of the UST, LERW, and of boundaries of macroscopic critical percolation clusters. He especially conjectured that the scaling limits are conformally invariant — this was also predicted earlier by physicists, who however had no tools to obtain rigorous constructions of the scaling limits. The proof of Schramm's conjectures was later established by Lawler, Schramm and Werner [LSW04] for the UST and LERW, and Smirnov [Smi01] for one percolation interface. Nevertheless, already in 1999, assuming the scaling limit exists, Schramm studied its properties. He noticed that there is a one parameter family $(\mathrm{SLE}_\kappa)_{\kappa\geq 0}$ of processes that are conformally invariant and satisfy the domain Markov property, a natural requirement from the scaling limits of interfaces — recall Section 2.4.

Schramm's SLE curves have also been shown to describe scaling limits of interfaces appearing in e.g. the critical Ising and random cluster models [CDCH$^+$14], and the discrete Gaussian free field [SS05, SS09, SS13]. However, there are models, such as the self-avoiding walk, for which a conformally invariant scaling limit has not been proven to exist (conjecturally, the SAW converges to $\mathrm{SLE}_{8/3}$). Also many variants of SLEs, which should describe scaling limits of more general curves, loops, level lines, etc., have been introduced, see e.g. [SW05, She09, MS13, KS15]. In [B], we consider multiple SLEs, candidates for scaling limits of multiple interfaces between boundary points, as in Figure 2.5. Results about the convergence of critical Ising interfaces to multiple SLE_3 curves have been obtained in [Izy11, CS12].

In this section, we discuss scaling limits of discrete interfaces and introduce the chordal SLE_κ and some of its variants. For a more extensive survey and related topics, we refer to e.g. [Wer03, Law05, Kem16].

3.1. **Conformal maps on the plane.** Conformal mappings preserve the local geometry — locally, angles remain unchanged. In this thesis, we consider the planar case, where conformal maps are also sometimes called univalent. For references about complex analysis, we recommend e.g. [Ahl79, Pom92].

A non-empty open connected subset $\Lambda \subset \mathbb{C}$ of the complex plane $\mathbb{C}$ is called a domain. If also the complement $(\mathbb{C} \cup \{\infty\}) \setminus \Lambda$ is connected, Λ is said to be simply connected. A mapping $f\colon \Lambda \to \mathbb{C}$ is said to be conformal if it is analytic and one-to-one. Locally, around a fixed point $z_0 \in \Lambda$, a conformal map looks like a scaling composed with a rotation and translation: f has non-vanishing derivative $f'(z_0) \neq 0$, and therefore its Taylor expansion up to first order is

$$f(z) \approx f(z_0) + f'(z_0) \times (z - z_0), \tag{3.1}$$

where $f(z_0)$ gives the translation, $\frac{f'(z_0)}{|f'(z_0)|}$ the rotation, and $|f'(z_0)|$ the scaling.

The Riemann mapping theorem states that any two simply connected domains $\Lambda, \Lambda' \subsetneq \mathbb{C}$ are conformally equivalent, and the conformal bijection $\phi \colon \Lambda \to \Lambda'$ between them is unique if one fixes three real parameters: the image of a chosen interior point of Λ and the argument of the derivative of ϕ at that point. For Jordan domains, that is, simply connected domains Λ, Λ' whose boundaries $\partial\Lambda, \partial\Lambda'$ are homeomorphic to the circle, we have the following generalization, see e.g. [Pom92, Corollary 2.7][3].

Theorem (Riemann mapping theorem)**.** *Let $\Lambda, \Lambda' \subsetneq \mathbb{C}$ be Jordan domains and $z_1, z_2, z_3 \in \partial\Lambda$ and $w_1, w_2, w_3 \in \partial\Lambda'$ points appearing in counterclockwise order along the boundary. Then there exists a unique conformal bijection $\phi \colon \Lambda \to \Lambda'$ such that $\phi(z_i) = w_i$, for $i = 1, 2, 3$.*

Conformal self-maps of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im\mathfrak{m}(z) > 0\}$ form the group $\mathrm{PSL}(2, \mathbb{R})$ acting as Möbius transformations $\mu(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In the spirit of the Riemann mapping theorem, to specify a Möbius map, it suffices to fix the images of three points. On the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, the maps μ act as fractional linear transformations with $\mu(\infty) = a/c$ and $\mu(-d/c) = \infty$.

3.2. **Brownian motion.** A classical example of a scaling limit of a discrete process is the standard Brownian motion (BM). In one dimension, it is the stochastic process $(B_t)_{t \geq 0}$ such that $B_0 = 0$ and

- the increments $B_{t_1} - B_{s_1}, \ldots, B_{t_n} - B_{s_n}$ are independent for any $0 \leq s_1 < t_1 \leq \cdots \leq s_n < t_n$,
- for $t > 0$, the increments $B_{t+s} - B_s$ have the normal distribution $N(0, t)$, independently of $s \geq 0$,
- and the map $t \mapsto B_t$ is almost surely continuous.

A standard d-dimensional BM is a process $(B_t^{(1)}, \ldots, B_t^{(d)})_{t \geq 0}$ where all components are independent standard one-dimensional Brownian motions. By Donsker's theorem, it is the scaling limit of the symmetric random walk on $\mathbb{Z}^d$, see e.g. [LL10, Theorem 3.5.1][4]. Interestingly, the two-dimensional BM is conformally invariant [Law05, Theorem 2.2], so the SRW indeed has a conformally invariant scaling limit.

3.3. **Chordal SLE$_\kappa$.** Random curves describing scaling limits of critical interfaces are expected to have two natural properties: conformal invariance (CI) and the domain Markov property (DMP), stated below. In the underlying physical models, (DMP) is already present in the discrete level (recall Section 2.4), and (CI) is conjectured to hold because of the scale-invariance at criticality (recall Section 2.6). To formally state these two requirements, we consider a random oriented curve $\gamma \colon [0,1] \to \overline{\Lambda}$, with law $\mathsf{P}^{(\Lambda;\xi,\eta)}$, connecting two distinct boundary points $\xi, \eta \in \partial\Lambda$ of a simply connected planar domain Λ, see Figure 3.1. We have chosen a particular paramerization for γ, with $\gamma(0) = \xi$ and $\gamma(1) = \eta$, but one has to bear in mind that the probability measure $\mathsf{P}^{(\Lambda;\xi,\eta)}$ of γ is in fact defined on a space $\mathcal{M}$ of curves modulo increasing reparametrizations — see Section 3.4 for more details. In order for γ to describe the scaling limit of a discrete interface, we require that the family $\big(\mathsf{P}^{(\Lambda;\xi,\eta)}\big)$ of probability measures satisfies:

(CI) **Conformal invariance:** If $\phi \colon \Lambda \to \Lambda'$ is a conformal map, and γ has the law $\mathsf{P}^{(\Lambda;\xi,\eta)}$, then its image $\phi \circ \gamma$ has the law $\phi_* \mathsf{P}^{(\Lambda;\xi,\eta)} = \mathsf{P}^{(\phi(\Lambda);\phi(\xi),\phi(\eta))}$, where ϕ_* is the pushforward of ϕ.

(DMP) **Domain Markov property:** Let τ be a stopping time with respect to the random curve γ. Conditionally on an initial segment $\gamma|_{[0,\tau]}$, the remaining part of γ has the law $\mathsf{P}^{(\Lambda';\gamma(\tau),\eta)}$, where Λ' is the component of the domain $\Lambda \setminus \gamma[0,\tau]$ containing η on its boundary.

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[3] A similar statement holds true for more general domains as well, replacing the boundary points by prime ends [Pom92].

[4] In dimensions $d \geq 4$ (and $d = 1$), BM is the scaling limit of the LERW as well [Law91]. However, the scaling limit of the two-dimensional LERW is not a BM — it is the SLE$_2$ [LSW04]. For $d = 3$, the scaling limit of the LERW is known to be invariant under rotations and scalings [Koz07] but the process is not well understood.

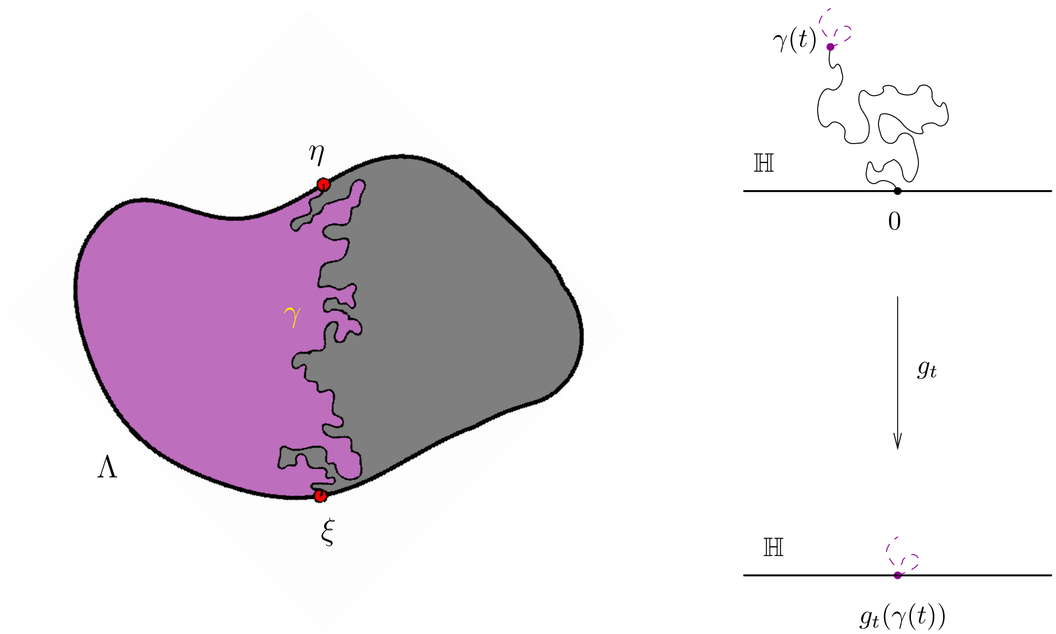

FIGURE 3.1. The left figure depicts a curve γ between two boundary points $\xi, \eta \in \partial\Lambda$ of a simply connected domain Λ. The figure on the right illustrates the Loewner map $g_t \colon \mathbb{H} \setminus K_t \to \mathbb{H}$. The domain Markov property (DMP) states that the purple piece of the curve in the lower picture is distributed as the chordal SLE_κ in $\mathbb{H}$ from $g_t(\gamma(t)) = X_t \in \mathbb{R}$ to ∞. The image of the tip of the SLE_κ curve γ is the driving process $X_t = \sqrt{\kappa} B_t$.

Schramm made the crucial observation that the random curves γ with the law $\mathsf{P}^{(\Lambda;\xi,\eta)}$ satisfying the two properties (CI) and (DMP) form a one-parameter family [Sch00, RS05]. In his seminal paper [Sch00], he introduced this process (which he called the stochastic Loewner evolution), $(\mathrm{SLE}_\kappa)_{\kappa \geq 0}$, as a random growth process arising from the Loewner differential equation (3.2), as explained below. It was later proved by Rohde and Schramm [RS05] that the SLE_κ process indeed defines an a.s. continuous random curve[5]. It is a non-self-intersecting fractal curve, with Hausdorff dimension $\kappa/8+1$ for $0 \leq \kappa \leq 8$ [Bef08] (and dimension 2 for $\kappa \geq 8$). It is a.s. a simple curve when $0 \leq \kappa \leq 4$, space filling when $\kappa \geq 8$, and neither when $4 < \kappa < 8$. In this thesis, we consider the parameter range $\kappa \in (0, 8)$.

According to the Riemann mapping theorem, for any simply connected domain Λ with two boundary points $\xi, \eta \in \partial\Lambda$, there exists a conformal bijection $\phi \colon \Lambda \to \mathbb{H}$ such that $\phi(\xi) = 0$ and $\phi(\eta) = \infty$ (if Λ is not a Jordan domain, one uses prime ends ξ, η). By (CI), it therefore suffices to construct the SLE_κ curve in the upper half-plane $\mathbb{H}$ from 0 to ∞. In its construction as a growth process, the time evolution of the SLE_κ curve is encoded in a solution to the Loewner differential equation: a collection $(g_t)_{t \geq 0}$ of conformal maps $z \mapsto g_t(z)$. Such maps were first considered by Charles Loewner in the 1920s while studying the Bieberbach conjecture [Loe23]. He managed to describe certain growth processes by a single ordinary differential equation, the Loewner equation. In the upper-half plane $\mathbb{H}$, it has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\, g_t(z) = \frac{2}{g_t(z) - X_t}, \qquad g_0(z) = z, \tag{3.2}$$

for $z \in \mathbb{H}$ and $t \geq 0$, where X_t is a real valued function, called the driving function. The solution of (3.2) is defined up to the explosion time $T_z = \inf\{t \geq 0 \mid g_t(z) = X_t\}$, and if K_t denotes the closure of the set $\{z \in \mathbb{H} \mid T_z < t\}$, then $(K_t)_{t \geq 0}$ defines a growth process. For each $t \in [0, T_z)$, the map $z \mapsto g_t(z)$ is the unique conformal isomorphism $g_t \colon \mathbb{H} \setminus K_t \to \mathbb{H}$ normalized such that $g_t(z) = z + o(1)$ as $z \to \infty$.

[5]To be precise, for $\kappa = 8$ this was shown in [LSW04].

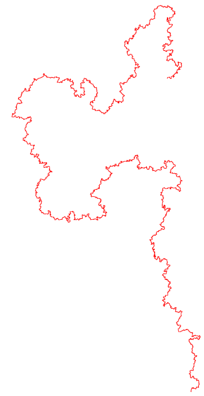

FIGURE 3.2. Simulation of the LERW on a 2000×2000 square grid, started from an interior point (top) and stopped at the boundary (bottom). Locally, the scaling limit of this curve looks like the chordal SLE_2, whose Hausdorff dimension is $5/4$.

Originally, Loewner considered continuous, deterministic driving functions (continuity of X_t ensures that K_t grow only locally). The groundbreaking idea of Schramm was to make X_t a random driving process: let $X_t = \sqrt{\kappa} B_t$ be a standard BM of speed κ. Then, the growth process $(K_t)_{t \geq 0}$ is generated by a random curve $\gamma \colon [0, \infty) \to \overline{\mathbb{H}}$ in the sense that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. This curve is (a parametrization of) the chordal SLE_κ in $\mathbb{H}$ from 0 to ∞, and K_t is its hull, see also Figure 3.1.

3.4. **Convergence of critical lattice models.** A step towards the conformal invariance of the scaling limit of a critical lattice model is to show that all interfaces and correlations of the model converge to a conformally invariant (covariant) scaling limit. For instance, the convergence of the correlations of the planar Ising model has been established in [Hon10, HS13, CI13, CHI15], and of the interfaces in [CDCH+14, Izy16]. Of course, one has to specify the topology with respect to which the convergence is defined. The discrete approximations $\mathcal{G}_\delta$ of continuous domains Λ are usually taken to converge in the Carathéodory topology, see [Pom92]. Observables of the models on $\mathcal{G}_\delta$ should converge to conformally covariant functions on Λ uniformly on compact subsets of Λ and with respect to the approximations $\mathcal{G}_\delta$, see e.g. [Smi10b, CS11, CDCH+14]. The convergence of the random interfaces is defined in terms of the weak convergence[6] of their probability measures to the law of a continuum random curve (the interfaces are considered as equivalence classes of curves in a suitable metric space $\mathcal{M}$, see below). When studying geometric features of the curves, it is important to establish the convergence in this strong topology.

In this thesis, we consider interfaces between boundary points, expected to converge to the chordal SLE_κ. Such interfaces γ_δ emerge from boundary points ξ_δ, η_δ where the boundary conditions change, as in Figures 2.3 and 3.1. For each triple $(\Lambda; \xi, \eta)$, we consider the curve γ as an oriented curve in Λ between the boundary points $\xi, \eta \in \partial\Lambda$ modulo increasing reparametrizations. We define the metric [AB99]

$$d(\gamma^{(1)}, \gamma^{(2)}) := \inf \Big\{ \big|\big|\gamma^{(1)} \circ \phi_1 - \gamma^{(2)} \circ \phi_2\big|\big|_\infty \;\Big|\; \phi_1, \phi_2 \colon [0,1] \to [0,1] \text{ are increasing bijections} \Big\} \tag{3.3}$$

on the space $\mathcal{M}$ of equivalence classes $[\gamma]$ of curves (i.e., continuous maps) $\gamma \colon [0,1] \to \mathbb{C}$, where the equivalence relation is given by the identification of curves with zero distance: $\big[\gamma^{(1)}\big] = \big[\gamma^{(2)}\big]$ if and only if $d(\gamma^{(1)}, \gamma^{(2)}) = 0$. We abuse the notation and terminology by choosing some representative and denoting its equivalence class by $\gamma = [\gamma]$. Each random curve γ_δ is distributed according to a probability measure $\mathbb{P}_\delta = \mathbb{P}^{(\mathcal{G}_\delta; \xi_\delta, \eta_\delta)}$, which is determined by the underlying lattice model. The measure $\mathbb{P}_\delta$ is supported on the equivalence classes of curves $\gamma_\delta \colon [0,1] \to \mathcal{G}_\delta$ such that $\gamma_\delta(0) = \xi_\delta$ and $\gamma_\delta(1) = \eta_\delta$.

[6]A sequence $(\mathbb{P}_n)_{n \in \mathbb{N}}$ of probability measures on a metric space $\mathcal{M}$ converges weakly to a probability measure $\mathbb{P}$ on $\mathcal{M}$ if for any continuous, bounded function $f \colon \mathcal{M} \to \mathbb{R}$, the expectations converge: $\mathbb{E}_n[f] \to \mathbb{E}[f]$ as $n \to \infty$.

3.5. **Scaling limit of the loop-erased random walk is SLE_2.** As an example of the convergence of discrete curves to the chordal SLE_κ, we consider the LERW. Let Λ be a bounded simply connected domain with two boundary points $\xi, \eta \in \partial\Lambda$, and let $\mathsf{P} = \mathsf{P}^{(\Lambda;\xi,\eta)}$ be the law of the chordal SLE_2. For small $\delta > 0$, let $\Lambda_\delta \subsetneq \mathbb{C}$ be a simply connected domain whose boundary is a path on the square grid $\delta\mathbb{Z}^2$ of mesh δ, and let $\xi_\delta, \eta_\delta \in \partial\Lambda_\delta$. Suppose that $(\Lambda_\delta;\xi_\delta,\eta_\delta)$ converges to $(\Lambda;\xi,\eta)$ in the Carathéodory sense[7]: there exist conformal bijections $\phi_\delta \colon \mathbb{D} \to \Lambda_\delta$ and $\phi \colon \mathbb{D} \to \Lambda$, from the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, such that $\phi(-1) = \xi$, $\phi(1) = \eta$, $\phi_\delta(-1) = \xi_\delta$, $\phi_\delta(1) = \eta_\delta$, and $\phi_\delta \to \phi$ as $\delta \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $\mathcal{G}_\delta$ be the square grid with vertices $\delta\mathbb{Z}^2 \cap \overline{\Lambda_\delta}$, and with those egdes of $\delta\mathbb{Z}^2 \cap \overline{\Lambda_\delta}$ which intersect Λ_δ. Consider the LERW on the grid $\mathcal{G}_\delta$ from ξ_δ conditioned to exit $\mathcal{G}_\delta$ through η_δ (after its first step from ξ_δ into Λ_δ) and denote the probability measure of this random curve by $\mathbb{P}_\delta = \mathbb{P}^{(\mathcal{G}_\delta;\xi_\delta,\eta_\delta)}$.

Theorem. [LSW04, Zha08] *The measures $\mathbb{P}_\delta$ converge weakly to P as $\delta \to 0$ with respect to the metric* (3.3) *on the space $\mathcal{M}$ of equivalence classes of curves.*

Originally, in [LSW04], the proof of the convergence was established first for the driving terms and then strengthened to the stronger convergence on $\mathcal{M}$. Lawler, Schramm and Werner treated in [LSW04] the radial SLE_2, running from a boundary point to an interior point. Dapeng Zhan proved in [Zha08] the chordal case and some generalizations. We give below a brief idea for a proof of the convergence directly with respect to the metric (3.3) on the space $\mathcal{M}$, based on ideas presented in [LSW04, KS12, CDCH$^+$14].

Idea of proof. By [KS12, Theorem 4.18], the LERW satisfies a certain geometric condition (C2) concerning estimating the probability of crossings of topological quadrilaterals (equivalent to a condition called (G2) which concerns annulus crossings). It therefore follows from [KS12, Theorem 1.5] that the sequence $(\mathbb{P}_\delta)_{\delta>0}$ of probability measures is tight, which implies that there exist subsequences converging weakly as $\delta \to 0$ with respect to the metric (3.3), by Prohorov's theorem, see e.g. [Bil99, Theorem 5.1].

Let Q be the law of a limiting curve γ of such a subsequence of LERWs. To show the uniqueness of the limit $[\gamma] \in \mathcal{M}$, one identifies the measure Q with the law P of the chordal SLE_2. From the condition (C2) it also follows by [KS12, Theorem 1.5] that the limiting curve γ is described by the Loewner equation (3.2) with a random driving process $(X_t)_{t\geq 0}$. One therefore only has to show that the driving process is that of the chordal SLE_2, that is, $X_t = \sqrt{2}B_t$. This is established by considering a suitable discrete harmonic observable which converges to a harmonic function as $\delta \to 0$, see [LSW04]. It is important to note that to identify the limit, one has to use quite specific information about the model[8].

Remark. To establish the convergence in domains with a rough boundary, one regards the curves as equivalence classes of mappings $(0,1) \to \mathbb{C}$ with the convergence on compact subsets of $(0,1)$. This is necessary as in some simply connected domains, there might not exist curves starting from a given boundary point (nor a prime end). We refer to [KS12, Corollary 1.8] for more details.

3.6. **Variants of SLE_κ.** Different types of interfaces in lattice models have slightly different scaling limits. Besides the chordal SLE_κ, one is thus led to consider more general random curves, variants of SLE_κ. Usually, such curves are locally absolutely continuous with respect to the chordal SLE_κ (i.e., their segments are absolutely continuous), and therefore, the law of the variant can be described in terms of its Radon-Nikodym derivative[9] with respect to the chordal SLE_κ. An important consequence of the absolute continuity is that almost sure properties for the chordal SLE_κ also hold a.s. for the variants.

For instance, the chordal $\mathrm{SLE}_\kappa(\rho)$, for $\rho \in \mathbb{R}$, is a variant introduced in [LSW03], consisting of one curve γ evolving according to the Loewner equation (3.2) with the driving function W_t, and one force point[10]

[7] This convergence is essentially equivalent to the Carathéodory convergence, see [KS12].

[8] In general, the parameter $\kappa > 0$ of the SLE_κ should characterize the universality class of the physical model whose interfaces are to converge in the scaling limit to the SLE_κ.

[9] If $\mathbb{P}_1$ and $\mathbb{P}_2$ are two probability measures on the same probability space (Ω, Σ), then $\mathbb{P}_1$ is said to be absolutely continuous with respect to $\mathbb{P}_2$ if we have $\mathbb{P}_1[A] = 0$ whenever $\mathbb{P}_2[A] = 0$ and $A \in \Sigma$. In this case, there exists a measurable function $F = \frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}_2} \colon \Omega \to [0,\infty)$, called the Radon-Nikodym derivative, such that we have $\mathbb{P}_1[A] = \int_A F \mathrm{d}\mathbb{P}_2$ for all $A \in \Sigma$.

[10] The force point can be interpreted as giving rise to the appropriate boundary conditions in the corresponding statistical mechanics models, in terms of boundary changing operators in the limiting CFT, see e.g. [BB03a, Kyt06].

with time evolution Y_t, such that the pair (W_t, Y_t) satisfies the stochastic differential equations (SDEs)

$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \frac{\rho}{W_t - Y_t}\,\mathrm{d}t, \qquad \mathrm{d}Y_t = \frac{2}{Y_t - W_t}\,\mathrm{d}t, \qquad W_0 = x, \qquad Y_0 = y.$$

In other words, the time evolution $Y_t = g_t(y)$ of the force point is given by the solution g_t to the Loewner equation (3.2) with the driving function W_t, a Brownian motion of speed κ with the drift $\frac{\rho}{W_t - Y_t}$. In fact, initial segments of the $\mathrm{SLE}_\kappa(\rho)$ curve with the law $\mathsf{P}^{(\mathbb{H};x,\infty;y)}$ and the chordal SLE_κ curve with the law $\mathsf{P}^{(\mathbb{H};x,\infty)}$ are mutually absolutely continuous[11] [Wer04, Dub07b], with the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathsf{P}^{(\mathbb{H};x,\infty;y)}}{\mathrm{d}\mathsf{P}^{(\mathbb{H};x,\infty)}}\bigg|_{\mathcal{F}_t} = |g_t'(y)|^{\frac{\rho(\rho-\kappa+4)}{4\kappa}} \times \frac{\mathcal{Z}\big(g_t(\gamma(t)); g_t(y)\big)}{\mathcal{Z}(x;y)}, \tag{3.4}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by a Brownian motion under the measure $\mathsf{P}^{(\mathbb{H};x,\infty)}$, and $\mathcal{Z}(x;y) = |y-x|^{\rho/\kappa}$ is a partition function[12], which thus characterizes the $\mathrm{SLE}_\kappa(\rho)$ process.

Remark. The Radon-Nikodym derivative (3.4) is in fact the Girsanov density of the probability measure $\mathsf{P}^{(\mathbb{H};x,\infty;y)}$ with respect to the probability measure $\mathsf{P}^{(\mathbb{H};x,\infty)}$. By Girsanov's theorem — see e.g. [RY05, Chapter 8] — the change of measure is established using the positive local martingale $M_t = |g_t'(y)|^{\frac{\rho(\rho-\kappa+4)}{4\kappa}} \times \mathcal{Z}(g_t(\gamma(t)); g_t(y))$ with respect to the chordal SLE_κ probability measure $\mathsf{P}^{(\mathbb{H};x,\infty)}$.

3.7. **Multiple SLEs.** Several interfaces connecting distinct boundary points in two-dimensional critical lattice models are expected to converge to conformally invariant interacting random curves, multiple SLEs — see Figures 2.5 and 3.3. For the Ising model, the convergence has been proved in [CS12, Izy16].

Multiple SLEs are processes of $2N$ non-crossing random curves started from given boundary points $x_1, \ldots, x_{2N}$ of a simply connected domain. Their law should be conformally invariant, so without loss of generality, we consider the upper half-plane $\mathbb{H}$ with $x_1 < \cdots < x_{2N}$. As a growth process, a multiple SLE_κ is described by a multi-slit Loewner equation, with the driving processes $X_t^{(1)}, \ldots, X_t^{(2N)}$ of the $2N$ curves that describe their time evolution. These processes satisfy the SDEs [BBK05, Gra07]

$$\mathrm{d}X_t^{(j)} = \sqrt{\kappa}\,\mathrm{d}B_t^{(j)} + \kappa\,\partial_j \log \mathcal{Z}(X_t^{(1)}, \ldots, X_t^{(2N)})\,\mathrm{d}t + \sum_{i\neq j} \frac{2}{X_t^{(j)} - X_t^{(i)}}\,\mathrm{d}t, \qquad j = 1, \ldots, 2N, \tag{3.5}$$

involving a partition function $\mathcal{Z}(x_1, \ldots, x_{2N})$ in the drift (defined only before the curves meet, though).

For each $j = 1, \ldots, 2N$, the initial segment of the j:th curve $\gamma^{(j)}$ started from x_j is absolutely continuous with respect to the chordal SLE_κ from x_j to ∞, with Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathsf{P}^{(\mathbb{H};x_j)}_{\gamma^{(j)}}}{\mathrm{d}\mathsf{P}^{(\mathbb{H};x_j,\infty)}}\bigg|_{\mathcal{F}_t} = \prod_{i\neq j} |g_t'(x_i)|^{\frac{6-\kappa}{2\kappa}} \times \frac{\mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), g_t(\gamma^{(j)}(t)), g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)}{\mathcal{Z}(x_1, \ldots, x_{2N})}, \tag{3.6}$$

where $g_t \colon H_t^{(j)} \to \mathbb{H}$ is the unique conformal isomorphism from the unbounded component $H_t^{(j)}$ of $\mathbb{H} \setminus \gamma^{(j)}[0,t]$ to the upper-half plane $\mathbb{H}$ such that $g_t(z) = z + o(1)$ as $z \to \infty$.

It is natural to require that the time parametrization of the curves does not matter. This reparametrization invariance yields a system of PDEs of second order that the partition function $\mathcal{Z}$ has to satisfy[13], see (8.5) in Section 8. To find the PDEs, Julien Dubédat derived in [Dub07a] commutation relations for the infinitesimal generators of the driving processes $X_t^{(1)}, \ldots, X_t^{(2N)}$ of the curves, from the reparametrization invariance assumption. The rough idea is to consider two of the curves and grow them in two different orders — first one and then the other, or vice versa. Dubédat's commutation relation arises from comparing the expectations of test functions obtained by both ways of growing, see [Dub07a].

[11] The absolute continuity holds until we have $W_t = Y_t$, or Y_0 is disconnected from ∞ by the curve.

[12] See also [Kyt06] for a CFT interpretation of the partition function.

[13] One can also find this system of PDEs for all $j = 1, \ldots, 2N$ from the vanishing drift terms of the local martingales $M_t^{(j)} = \prod_{i\neq j} |g_t'(x_i)|^{\frac{6-\kappa}{2\kappa}} \times \mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), g_t(\gamma^{(j)}(t)), g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)$ with respect to $\mathsf{P}^{(\mathbb{H};x_j,\infty)}$, which appear in the Radon-Nikodym derivatives (i.e., Girsanov densities) of Equation (3.6). In the calculation, one applies Itô's formula similarly as in the next section, see also [BBK05].

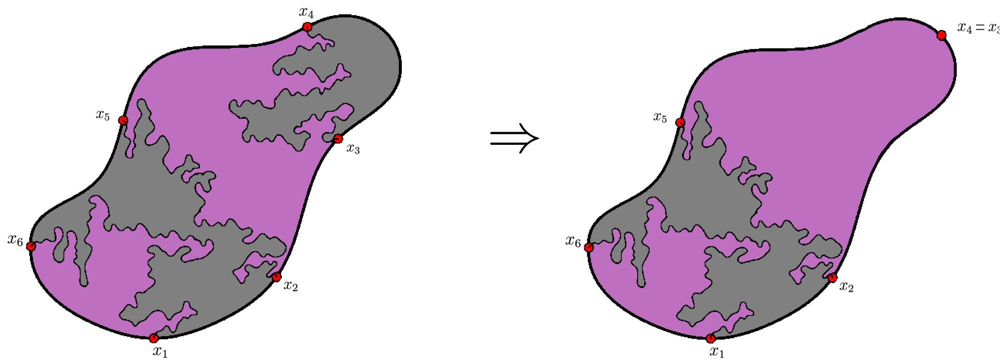

FIGURE 3.3. Schematic illustration of a multiple SLE_κ. Taking the boundary points x_3 and x_4 together, the corresponding curve is removed. The marginal law of the remaining curves should be given by a multiple SLE_κ with one curve less. See also Equation (8.7) in Section 8 for the interpretation in terms of partition functions $\mathcal{Z}_\alpha$.

By the conformal invariance of the law of the random curves, the partition function $\mathcal{Z}$ should also have a conformal covariance property, given in Equation (8.6) in Section 8. Finally, $\mathcal{Z}$ must be a positive function, by the SDE (3.5). In fact[13], the Radon-Nikodym derivative (3.6) is again a Girsanov density with a positive local martingale involving the partition function $\mathcal{Z}$.

3.8. **Multiple SLE pure geometries.** Multiple SLEs satisfy the two properties (CI) and (DMP), the latter adjusted to the situation of several curves (see [B] for details). However, the attempt to classify such processes as a one-parameter family, in the spirit of [Sch00], fails due to the presence of nontrivial conformal moduli for the domain with at least four marked boundary points: using the Riemann mapping theorem, we may only fix the images of three points under a conformal isomorphism between two simply connected domains. Indeed, for a fixed $\kappa > 0$, the multiple SLE_κ probability measures should form a convex set. Bauer, Bernard and Kytölä suggested in [BBK05] that the extremal points ("pure measures", or "pure geometries") of this convex set should be processes which have a deterministic pairwise connectivity of the $2N$ boundary points by the N non-crossing curves, as in Figure 2.5. To these extremal processes, one associates pure partition functions $\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})$, labeled by the possible connectivities α (i.e., planar pair partitions of $2N$ points). The functions $\mathcal{Z}_\alpha$ should be specified by certain recursive boundary conditions given in terms of asymptotics when any two variables x_j, x_{j+1} tend to a common limit, see Equation (8.7), and Figure 3.3 for an illustration. The contribution of this thesis to finding the multiple SLE pure partition functions $\mathcal{Z}_\alpha$ is discussed in Section 8, see also [B].

3.9. **Green's functions for the chordal SLE_κ.** Estimates concerning the so called SLE_κ Green's functions have been used in studying the fractal properties of the curve, such as its Hausdorff dimension [RS05, Bef08] and Minkowski content [AS08, AS11, Law15, LR15]. The problem of finding explicit expressions for the SLE_κ boundary Green's functions was studied in [JJK16, B], see also Section 8.2.

Let $\mathsf{P}^{(\mathbb{H};x,\infty)}$ be the law of the chordal SLE_κ curve γ in $\mathbb{H}$ from $x \in \mathbb{R}$ to ∞. A natural question concerning the behavior of the curve is the probability of the event that γ visits given points. The probability of visiting a single point is zero, but a non-zero suitably renormalized limit of the probability that γ visits a small neighborhood of a point exists [Law09, Law15, LR15] — the renormalization factor is a power of the size of the neighborhood, and the exponent is known in physics as the critical exponent associated to the corresponding statistical mechanics model, with the universality class specified by κ.

Already in 2005 [RS05], Rohde and Schramm established the upper bound $1 + \kappa/8$ for the Hausdorff dimension of the SLE_κ curve, assuming the existence of the renormalized limit

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\kappa/8-1} \, \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\mathrm{dist}(\gamma, z) < \varepsilon\big] \;=:\; \text{const.} \times G(x; z), \tag{3.7}$$

for any bulk point $z \in \mathbb{H}$. The limit (3.7) was only recently shown to exist by Lawler and Rezaei [LR15]. The function G is called the SLE_κ Green's function, and it is defined only up to a multiplicative constant[14]. Nevertheless, the Green's function is well understood. It has the following explicit form [RS05]:

$$G(x;z) \;\propto\; \Im\mathfrak{m}(z-x)^{\kappa/8-1}\big(\sin\arg(z-x)\big)^{8/\kappa-1}.$$

For a boundary point $y \in \mathbb{R}$, the SLE_κ boundary Green's function ζ is defined with a different scaling[15],

$$\lim_{\varepsilon\searrow 0} \varepsilon^{1-8/\kappa}\, \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\mathrm{dist}(\gamma,y)<\varepsilon\big] \;=:\; \text{const.}\times\zeta(x;y). \tag{3.8}$$

The existence of the limit (3.8) was also established by Lawler in [Law15]. The scale-invariance of the SLE_κ shows that the Green's function has the simple form[16] $\zeta(x;y) \propto |y-x|^{1-8/\kappa}$.

One similarly defines the N-point Green's functions. For boundary points $y_1,\ldots,y_N \in \mathbb{R}$, we have

$$\lim_{\varepsilon_1,\ldots,\varepsilon_N\searrow 0} (\varepsilon_1\cdots\varepsilon_N)^{1-8/\kappa}\, \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\mathrm{dist}(\gamma,y_1)<\varepsilon_1,\ldots,\mathrm{dist}(\gamma,y_N)<\varepsilon_N\big] \;=:\; \text{const.}\times\zeta(x;y_1,\ldots,y_N).$$

In a general simply connected domain Λ, the boundary N-point Green's function for SLE_κ from ξ to η is determined by the following conformal covariance property:

$$\zeta_{(\Lambda;\xi,\eta)}(y_1,\ldots,y_N) \;=\; \prod_{i=1}^N |\phi'(y_i)|^{8/\kappa-1}\times\zeta\big(x;\phi(y_1),\ldots,\phi(y_N)\big), \tag{3.9}$$

where $\phi\colon \Lambda\to\mathbb{H}$ is a conformal map with $\phi(\xi)=x$ and $\phi(\eta)=\infty$, and the points $y_1,\ldots,y_N \in \partial\Lambda$ are assumed to lie in a smooth part of the boundary, so that ϕ extends to their vicinity. To see why (3.9) should hold, note that an ε-neighborhood of a point z in $\overline{\Lambda}$ is mapped, roughly, to a neighborhood of the point $\phi(z)$ of size $\varepsilon\times|\phi'(z)|$. By the conformal invariance of the chordal SLE_κ, the limit of the renormalized probability of the event that the SLE_κ curve visits ε-neighborhoods of given points in Λ should thus be proportional to the right hand side of (3.9).

Remark. In the special case $\rho=\kappa-8$, the $\mathrm{SLE}_\kappa(\kappa-8)$ process is equivalent to the chordal SLE_κ conditioned to hit a given point $y=Y_0\in\mathbb{R}$, and the partition function appearing in the Radon-Nikodym derivative (3.4) is in fact the Green's function, that is, $\zeta(x;y)\propto|y-x|^{1-8/\kappa}=\mathcal{Z}(x;y)$.

Remark. Explicit formulas for the Green's functions are currently known only for the one point functions (3.7) and (3.8), and for the boundary two point function $\zeta(x;y_1,y_2)$ in terms of a hypergeometric function, see [SZ10]. To find expressions for them, one can use a general technique, which we briefly summarize in the next section. In relation with this thesis, this idea was exploited in the articles [JJK16, B] to find proposed formulas for the boundary Green's functions for any number N of points. The proof that the found formulas indeed are the Green's functions $\zeta(x;y_1,\ldots,y_N)$ is left for future work.

4. Partial differential equations from Itô calculus

To find explicit expressions for SLE observables, such as Green's functions and partition functions, one can exploit a common technique, which we shall briefly explain in this section. We begin with a brief summary about stochastic calculus à la Kiyoshi Itô and introduce most of the needed concepts. For more background on stochastic analysis and Itô calculus, we refer to textbooks, e.g., [Dur96, RW00a, RW00b, Law05, RY05, Dur10]. Then, in Section 4.3, we explain the technique to find SLE observables with an example application related to the chordal SLE_κ boundary Green's functions.

A crucial ingredient for us is an ordinary differential equation (ODE), or, more generally, a PDE system, derived from the knowledge that certain observables are local martingales with respect to the SLE_κ

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[14] To the author's knowledge, the constant in (3.7) is currently known explicitly only in the case when $\mathrm{dist}(\gamma,z)$ is replaced by conformal distance [Law09, Law15].

[15] The critical exponents are, in general, different for bulk points and boundary points. They are expected to be universal, depending only on the parameter κ associated to the model, and not e.g. on the microscopic details.

[16] As in the bulk case, to the author's knowledge, the multiplicative constant has been calculated explicitly only for the variant of the limit (3.8) with the conformal distance instead of the Euclidean one [Law15].

probability measure. Imposing suitable boundary conditions, one can hope to uniquely specify a solution to this PDE boundary value problem. After finding a solution, one then shows a posteriori that this solution indeed is the desired function, by an optional stopping argument.

4.1. **Martingales.** A prototypical example of a martingale is the Brownian motion. Martingales are processes $(M_t)_{t\geq 0}$ such that, given the history up to time t, the conditional expectation of M observed at time $s \geq t$ equals the present value M_t. More precisely, the stochastic process $(M_t)_{t\geq 0}$ is a martingale if

(i) it is integrable: $\mathbb{E}|M_t| < \infty$ for all $t \geq 0$,
(ii) it is adapted: M_t is $\mathcal{F}_t$-measurable for all $t \geq 0$,
(iii) and it satisfies the martingale property $\mathbb{E}[M_s \mid \mathcal{F}_t] = M_t$ for all $s \geq t$,

where $(\mathcal{F}_t)_{t\geq 0}$ is the underlying filtration, that is, the history up to time t.

One often wants to relax the condition (i) to hold only for up to some time, say. This can be done by localization. Local martingales are defined as processes $(M_t)_{t\geq 0}$ for which there is an increasing sequence of stopping times τ_n such that $\tau_n \nearrow \infty$ as $n \nearrow \infty$ almost surely, and the stopped processes $(M_t^{\tau_n})_{t\geq 0}$, defined by $M_t^{\tau_n} := M_{\min(t,\tau_n)}$ if $\tau_n > 0$, and $M_t^0 := 0$, are martingales, i.e., they satisfy (i) – (iii).

The optional stopping theorem [Dur10, Theorem 4.7.4] is a common tool in proofs concerning the SLE. It states that under certain conditions, the martingale property (iii) holds for stopping times as well.

Theorem (Optional stopping)**.** *Let $(M_t)_{t\geq 0}$ be a continuous martingale and τ, σ almost surely finite stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Suppose $\sigma \leq \tau$. If the martingale $(M_t)_{t\geq 0}$ is uniformly integrable, that is, $\lim_{n\to\infty} \sup_{t\geq 0} \mathbb{E}\big[|M_t| \, \mathbb{1}_{|M_t|\geq n}\big] = 0$, then we have $\mathbb{E}[M_\tau \mid \mathcal{F}_\sigma] = M_\sigma$.*

Optional stopping can be applied for instance when $(M_t)_{t\geq 0}$ is uniformly bounded, as then it clearly is uniformly integrable. An example application will be given in Section 4.3, with $\sigma = 0$, and $\mathbb{E}[M_\tau] = M_0$.

4.2. **Itô calculus.** For a continuous function $F\colon \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ which is continuously differentiable at least once in the variable $t \in \mathbb{R}_{>0}$ and twice in $x \in \mathbb{R}$, we almost surely have the following Itô differential:

$$\mathrm{d}F(t, B_t) \;=\; \partial_t F(t, B_t)\,\mathrm{d}t \;+\; \partial_x F(t, B_t)\,\mathrm{d}B_t \;+\; \frac{1}{2}\partial_x^2 F(t, B_t)\,\mathrm{d}t.$$

The intuition behind this type of a truncating Taylor expansion is that one can think the differentials $\mathrm{d}t$ and $\mathrm{d}B_t$ of as satisfying the multiplication rules given by their quadratic variations:

$$\mathrm{d}t\,\mathrm{d}t \;=\; 0, \qquad \mathrm{d}t\,\mathrm{d}B_t \;=\; 0, \qquad \mathrm{d}B_t\,\mathrm{d}B_t \;=\; \mathrm{d}\langle B, B\rangle_t \;=\; \mathrm{d}t.$$

We need to apply Itô's formula to functions of several x-variables, and to more general stochastic processes than the Brownian motion. Let thus $(B_t^{(1)}, \ldots, B_t^{(n)})$ be a standard n-dimensional Brownian motion, and let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration. Itô's formula generalizes nicely for semimartingales with respect to the Brownian motion, that is, stochastic processes Y_t satisfying an SDE of the form

$$\mathrm{d}Y_t \;=\; F(t)\,\mathrm{d}t \;+\; \sum_{k=1}^{n} G_k(t)\,\mathrm{d}B_t^{(k)}, \tag{4.1}$$

where Y_0 is a $\mathcal{F}_0$-measurable random variable, $G_1, \ldots, G_n$ are locally square integrable functions adapted to $(\mathcal{F}_t)_{t\geq 0}$, and F is a Lebesgue-measurable function adapted to $(\mathcal{F}_t)_{t\geq 0}$ such that almost surely, we have $\int_0^t |F(s)|\mathrm{d}s < \infty$ for all $t \geq 0$. Amongst such processes, one can easily characterize the ones which are local martingales: $(Y_t)_{t\geq 0}$ is a local martingale if and only if its drift term vanishes, i.e., $F \equiv 0$.

For two semimartingales $Y_t^{(1)}, Y_t^{(2)}$, satisfying SDEs of the form (4.1), their covariation is defined as

$$\langle Y^{(1)}, Y^{(2)}\rangle_t \;:=\; \sum_{k,l} G_k^{(1)}(t)\, G_l^{(2)}(t)\; \langle B_t^{(k)} B_t^{(l)}\rangle_t \;=\; \sum_{k} G_k^{(1)}(t)\, G_k^{(2)}(t)\, t.$$

Itô's formula for semimartingales can now be written as follows, see e.g. [RW00b, Theorem (32.8)].

Theorem (Itô's formula)**.** *Let $Y_t^{(1)},\ldots,Y_t^{(N)}$ be semimartingales, and let $\psi \in \mathcal{C}^2(\mathbb{R}^N)$. Then, also $\psi(Y_t^{(1)},\ldots,Y_t^{(N)})$ is a semimartingale, and we almost surely have*

$$\mathrm{d}\psi(Y_t^{(1)},\ldots,Y_t^{(N)}) \;=\; \sum_{j=1}^N \partial_j\psi(Y_t^{(1)},\ldots,Y_t^{(N)})\,\mathrm{d}Y_t^{(j)} \;+\; \frac{1}{2}\sum_{i,j=1}^N \partial_i\partial_j\psi(Y_t^{(1)},\ldots,Y_t^{(N)})\,\mathrm{d}\langle Y^{(i)},Y^{(j)}\rangle_t$$

for all $t\geq 0$. Moreover, $\psi(Y_t^{(1)},\ldots,Y_t^{(N)})$ is a local martingale if and only if the drift vanishes:

$$\sum_{j=1}^N F^{(j)}(t)\,\partial_j\psi(Y_t^{(1)},\ldots,Y_t^{(N)}) \;+\; \frac{1}{2}\sum_{i,j=1}^N\sum_{k=1}^n G_k^{(i)}(t)\,G_k^{(j)}(t)\,\partial_i\partial_j\psi(Y_t^{(1)},\ldots,Y_t^{(N)}) \;\equiv\; 0.$$

4.3. **Interval hitting probability of the chordal SLE$_\kappa$.** We will see shortly that conditional expectations are almost trivially martingales, by the tower property[17]. This property has been used in constructing SLE$_\kappa$ martingale observables, see e.g. [BB04, Smi06], to obtain predictions as well as rigorous results about various features of SLE.

Let $x<y$ and consider the probability $\mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\big]$ of the event that the chordal SLE$_\kappa$ curve γ in $\mathbb{H}$ from x to ∞ hits the boundary interval $[y,y+\varepsilon]$, assuming that $\kappa\in(4,8)$, in order to have a non-zero hitting probability. The corresponding conditional probability is a local martingale: it is bounded, $\mathcal{F}_t$-measurable, and by the tower property, we have

$$\mathsf{E}^{(\mathbb{H};x,\infty)}\Big[\mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\mid\mathcal{F}_t\big]\;\Big|\;\mathcal{F}_s\Big] \;=\; \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\mid\mathcal{F}_s\big]$$

for any $s\leq t$ (smaller than the time when γ swallows y, that is, the first time when $y\in K_t$). On the other hand, by the properties (DMP) and (CI) of the SLE$_\kappa$, this local martingale can be written in the form

$$\begin{aligned} M_t(x;y,y+\varepsilon) \;&:=\; \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\mid\mathcal{F}_t\big] \\ &=\; \mathsf{P}^{(\mathbb{H}\setminus K_t;\gamma(t),\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\big] \\ &=\; \mathsf{P}^{(\mathbb{H};X_t,\infty)}\big[\gamma\cap[g_t(y),g_t(y+\varepsilon)]\neq\emptyset\big] \\ &=\; M_0\big(X_t;g_t(y),g_t(y+\varepsilon)\big). \end{aligned}$$

Moreover, conformal invariance allows us to perform a translation and a scaling, so that we in fact get

$$M_t(x;y,y+\varepsilon) \;=\; M_0\big(X_t;g_t(y),g_t(y+\varepsilon)\big) \;=\; M_0\left(0;\frac{g_t(y)-X_t}{g_t(y+\varepsilon)-X_t},1\right). \tag{4.2}$$

It is convenient to denote by $Z_t(z)=g_t(z)-X_t$, and $Y_t=\frac{Z_t(y)}{Z_t(y+\varepsilon)}$, and $F(z):=M_0(0;z,1)$. Then, the local martingale (4.2) equals $M_t=F(Y_t)$. We will next calculate its Itô differential.

Remark. The renormalized limit $\lim_{\varepsilon\to 0}\varepsilon^{1-8/\kappa}M_0(x;y,y+\varepsilon)$ is analogous to the definition (3.8) of the boundary Green's function $\zeta(x;y)$ in Section 3.9, replacing the ε-neighborhood of y by the interval $[y,y+\varepsilon]$. The Euclidean neighborhood, however, does not behave very well under conformal maps, but the interval does, and in particular, we can explicitly calculate the interval hitting probability.

From Itô's formula and the Loewner equation (3.2), we obtain $\mathrm{d}Z_t(z)=\frac{2\,\mathrm{d}t}{Z_t(z)}-\sqrt{\kappa}\,\mathrm{d}B_t$, so that $\mathrm{d}\langle Z(z),Z(w)\rangle_t=\kappa\,\mathrm{d}t$. For the process $Y_t=h\big(Z_t(y),Z_t(y+\varepsilon)\big)$ with $h(z,w)=\frac{z}{w}$, we therefore obtain

$$\mathrm{d}Y_t \;=\; \frac{1}{Z_t(y+\varepsilon)^2}\left((\kappa-2)Y_t+\frac{2}{Y_t}-\kappa\right)\mathrm{d}t \;+\; \frac{1}{Z_t(y+\varepsilon)}(Y_t-1)\sqrt{\kappa}\,\mathrm{d}B_t.$$

We may now calculate the drift of $\mathrm{d}F(Y_t)=F'(Y_t)\,\mathrm{d}Y_t+\frac{1}{2}F''(Y_t)\,\mathrm{d}\langle Y,Y\rangle_t$, assuming F is regular enough. Then, we first solve for F, and after that verify (by optional stopping, see below) that the solution indeed equals the desired function, defined as $F(z)=M_t(0;z,1)$, and has the correct regularity.

[17]The following property is usually referred to as the tower property. Let $\Sigma_1\subset\Sigma_2\subset\Sigma$ be sigma-algebras and let $X\in\mathcal{L}^1(\mathbb{P})$ be an absolutely integrable random variable with respect to the probability measure $\mathbb{P}$ on (Ω,Σ). Then we have $\mathbb{E}[\mathbb{E}[X\mid\Sigma_2]\mid\Sigma_1]=\mathbb{E}[X\mid\Sigma_1]$ almost surely.

To begin, since M_t is a local martingale, the Itô drift of $F(Y_t)$ should vanish. This gives the constraint

$$\frac{1}{Z_t(y+\varepsilon)^2}\Big(\Big((\kappa-2)Y_t + \frac{2}{Y_t} - \kappa\Big)F'(Y_t) + \frac{1}{2}(Y_t-1)^2\,\kappa\,F''(Y_t)\Big)\,\mathrm{d}t = 0.$$

Denoting by $z = Y_t$, we see that this results in the following ordinary differential equation (ODE) for F:

$$F''(z) + \frac{4-2(\kappa-2)z}{\kappa\, z(1-z)}\,F'(z) = 0. \tag{4.3}$$

On the other hand, F should be equal to the hitting probability M_t, wherefrom we find boundary conditions for F. First, if $y = x$ (i.e., $z = 0$), then the SLE_κ curve γ starting from x is to hit the interval $[x, x+\varepsilon]$. We thus expect that $F(0) = 1$. Second, because the probability that the SLE_κ hits a single point is zero, taking $\varepsilon = 0$ (i.e., $z = 1$), we get the condition $F(1) = 0$. Solving the ODE (4.3) with these boundary conditions, we obtain

$$F(z) = \frac{\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\,\Gamma(8/\kappa-1)}\int_z^1 u^{-4/\kappa}(1-u)^{8/\kappa-2}\,\mathrm{d}u,$$

and we also note that the integral is convergent when $\kappa \in (4,8)$. To show that this is the desired function in (4.2), we use optional stopping. Because the above F satisfies the ODE (4.3), the expression $F(Y_t)$ is a bounded local martingale. Let $\tau = T_y$ be the swallowing time of y by the SLE_κ curve. On the event $\{\gamma\cap[y,y+\varepsilon]=\emptyset\}$, we have $Y_\tau = 1$, and on the event $\{\gamma\cap[y,y+\varepsilon]\neq\emptyset\}$, we have $Y_\tau = 0$, almost surely. Using the boundary values $F(0) = 1$ and $F(1) = 0$, optional stopping now gives

$$\begin{aligned}
\mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\big] &= \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\big]\,F(0) + \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]=\emptyset\big]\,F(1)\\
&= \mathsf{E}^{(\mathbb{H};x,\infty)}\big[F(Y_\tau)\big]\\
&= F(Y_0) = F\Big(\frac{y-x}{y-x+\varepsilon}\Big)\\
&= \frac{\varepsilon^{8/\kappa-1}\,\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\,\Gamma(8/\kappa-1)}\int_0^1 (y-x+\varepsilon v)^{-4/\kappa}\Big(\frac{(1-v)^2}{y-x+\varepsilon}\Big)^{4/\kappa-1}\,\mathrm{d}v.
\end{aligned}$$

In particular, we have $\mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\big]\sim\varepsilon^{8/\kappa-1}$ as $\varepsilon\to 0$, and the limit is explicit:

$$\lim_{\varepsilon\to0}\varepsilon^{8/\kappa-1}\,\mathsf{P}^{(\mathbb{H};x,\infty)}\big[\gamma\cap[y,y+\varepsilon]\neq\emptyset\big] = \frac{\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\,\Gamma(8/\kappa-1)}\,\frac{\kappa}{8-\kappa}\,(y-x)^{1-8/\kappa}.$$

4.4. **Chordal SLE boundary visits.** One can similarly construct local martingales from other boundary visit probabilities. In terms of the N-point boundary Green's functions $\zeta(x;y_1,\ldots,y_N)$, the local martingale is [JJK16]

$$\prod_{i=1}^N\big|g_t'(y_i)\big|^{8/\kappa-1}\times\zeta\big(X_t;g_t(y_1),\ldots,g_t(y_N)\big) = \zeta_{(\mathbb{H}\setminus K_t;\gamma(t),\infty)}(y_1,\ldots,y_N). \tag{4.4}$$

It is straightforward to calculate the Itô differential of the local martingale (4.4), with the help of Itô's formula, the observation $g_t'(z) > 0$, and the SDEs

$$\mathrm{d}g_t(z) = \frac{2}{g_t(z)-X_t}\,\mathrm{d}t \qquad\text{and}\qquad \mathrm{d}g_t'(z) = -\frac{2g_t'(z)}{(g_t(z)-X_t)^2}\,\mathrm{d}t,$$

which follow from the Loewner equation (3.2). The vanishing drift of (4.4) gives the second order PDE

$$\Big[\frac{\kappa}{2}\frac{\partial^2}{\partial x^2} + \sum_{i=1}^N\Big(\frac{2}{y_i-x}\frac{\partial}{\partial y_i} - \frac{2(8/\kappa-1)}{(y_i-x)^2}\Big)\Big]\zeta(x;y_1,\ldots,y_N) = 0. \tag{4.5}$$

Thus, the local martingale (4.4), constructed using the tower property of conditional expectation and the defining properties (CI) and (DMP) of SLE_κ, yields a PDE for the Green's function ζ. In the case of three variables, this PDE together with the conformal covariance property (3.9) strongly restricts the form of the function $\zeta(x;y_1,y_2)$, because the translation and scaling covariance of ζ reduce (4.5) to an ODE

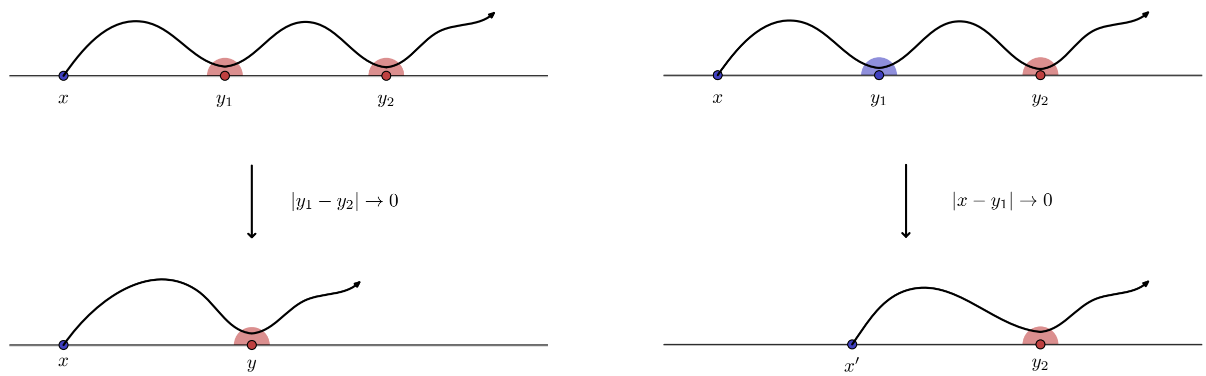


FIGURE 4.1. Illustration of the boundary conditions (4.6) for the two-point Green's function $\zeta(x; y_1, y_2)$ in the upper half-plane. Similar boundary conditions are used in the article, [JJK16, B] to find solutions $\zeta(x; y_1, y_2, \ldots, y_N)$ to the PDE (4.5) and the covariance property (3.9) — see also Section 8.2 for the contributions of this thesis.

(as in the case of the interval hitting probability). Imposing suitable boundary conditions, Schramm and Zhou found an explicit formula for $\zeta(x; y_1, y_2)$, in terms of a hypergeometric function [SZ10].

The boundary conditions for the function $\zeta(x; y_1, y_2)$ can be deduced from its definition as the renormalized probability of the event that the chordal SLE_κ curve γ from x to ∞ hits neighborhoods of y_1 and y_2. For example, when $x < y_1 < y_2$, we note that if x and y_1 are close, γ very likely immediately hits y_1, and then continues like the chordal SLE_κ from y_1 to ∞ (in the remaining domain). If instead y_1 and y_2 are close, γ either very likely hits both of them or neither of them, according to the probability to hit one point $y > x$ (see also Figure 4.1).

Heuristically, the following asymptotics should thus hold for the function $\zeta(x; y_1, y_2)$:

$$\begin{aligned} \lim_{y_1, y_2 \to y} |y_2 - y_1|^{8/\kappa - 1} \times \zeta(x; y_1, y_2) &= \zeta(x; y) \quad \propto |y - x|^{1 - 8/\kappa} \\ \lim_{x, y_1 \to x'} |y_1 - x|^{8/\kappa - 1} \times \zeta(x; y_1, y_2) &= \zeta(x'; y_2) \ \propto |y_2 - x'|^{1 - 8/\kappa}. \end{aligned} \tag{4.6}$$

This method allows one to find a candidate for the function $\zeta(x; y_1, y_2)$. It then remains to prove that the solution found in this way indeed is the desired Green's function, using optional stopping as before.

Remark. The PDE (4.5) together with the translation and scaling covariance (3.9) provide very powerful constraints for the two-point boundary Green's function. However, for the N-point functions depending on a larger number of points, the covariance properties are not sufficient to reduce the PDE (4.5) to just an ordinary differential equation, but one indeed has to solve partial differential equations. In fact, the Green's functions are also expected to satisfy PDEs of third order, by (non-rigorous) arguments from conformal field theory, described in Section 6.4 — see also the articles [JJK16, B, C][18].

For large N, the solution spaces to the PDE systems become large and substantially harder to study. In the articles [JJK16, B], proposals for formulas for the boundary N-point Green's functions are given. They are found by imposing asymptotic boundary conditions analogous to (4.6) — see also Section 8.2. The proof that these solutions indeed are the Green's functions remains for future work. The idea is a recursive optional stopping argument, see [JJK16, Section 5], and the main difficulty is to obtain sufficient control of the functions to establish uniform integrability of the local martingales.

[18] Interestingly, we prove in the article [C] that the scaling limits of boundary visit probabilities of the LERW in fact satisfy both the second order PDE (4.5) arising from stochastic calculus, and the third order PDEs predicted by conformal field theory, with $\kappa = 2$. These scaling limits should be proportional to the SLE_2 boundary Green's functions.

Part II: Algebraic techniques and conformal field theory

In this part, we summarize algebraic concepts related to the results of this thesis. Basically, in all the articles [A, B, C, D, E, F, G], we exploit the representation theory of a particular Hopf algebra, the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, to study questions in random geometry. In general, we consider solutions of PDE systems of conformal field theory, special cases of which are also present in the theory of SLEs.

We begin this part by introducing Hopf algebras: associative algebras with additional structure that guarantees useful properties for their representation theory — the representations of a Hopf algebra form a tensor category. We then discuss some aspects of conformal field theory (CFT) in Section 6, to motivate the topics of this thesis. There are many approaches to CFT, and we take the algebraic one.

In Section 7, we explain how to construct solutions to systems of PDEs known as Benoit & Saint-Aubin equations. These PDEs arise from the presence of singular vectors in representations of the Virasoro algebra in CFT. Interestingly, the PDEs (4.5) and (8.5), related to the SLE_κ boundary Green's functions and the multiple SLE_κ partition functions, respectively, arise as a special case. This reveals a link between SLE and CFT — indeed, there is evidence of a deep connection between these two, see e.g. [Car84, BB02, BB03a, FW03, Dub15a, Dub15b]. For instance, certain CFT correlation functions provide SLE local martingales, carrying an action of the Virasoro algebra[19], see [BB03b, Kyt07, Dub15a].

5. Hopf algebras and representation theory

In this section, we summarize some representation theory of Hopf algebras. Of particular interest to us are tensor product representations. Examples of Hopf algebras include quantum groups which, in the Drinfeld-Jimbo sense [Dri85, Dri87, Jim85], are q-deformations $\mathcal{U}_q(\mathfrak{g})$ of the enveloping algebras $\mathcal{U}(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$. Their definition was inspired by physics, where they arose as symmetries in exactly solvable two-dimensional lattice models. An important property of quantum groups is also that they can be regarded as braided Hopf algebras, which gives a natural tool to construct bimodules carrying both the action of the Hopf algebra and of the braid group. We briefly describe this in Section 5.8.

5.1. **Representations of associative algebras.** Let $\mathfrak{A}$ be an associative algebra over the field $\mathbb{C}$ of complex numbers, with the unit $1_{\mathfrak{A}} \in \mathfrak{A}$. We invite the reader to review the basic notions about the representation theory of $\mathfrak{A}$ from the literature, e.g. [Kas95, CR06, EGH+11]. In this thesis, we consider finite-dimensional representations of $\mathfrak{A}$, that is, linear maps ρ from $\mathfrak{A}$ to the space $\mathrm{End}(V)$ of endomorphisms of a finite-dimensional complex vector space V, respecting the algebra structure: $\rho(ab) = \rho(a)\rho(b)$. We recall that a representation $\rho\colon \mathfrak{A} \to \mathrm{End}(V)$ is called irreducible if its only $\mathfrak{A}$-invariant subspaces are V and $\{0\}$. Also, we say that a representation is completely reducible if it is isomorphic to a direct sum of finitely many irreducibles. By an $\mathfrak{A}$-intertwiner we mean a morphism of representations of $\mathfrak{A}$. Because the kernel and image of an intertwiner are also representations, the space $\mathrm{Hom}_{\mathfrak{A}}(V, W)$ of intertwiners between irreducible representations has a particularly simple structure.

Lemma (Schur's lemma)**.** *Let V and W be finite-dimensional irreducible representations of $\mathfrak{A}$. Then, we have* $\dim(\mathrm{Hom}_{\mathfrak{A}}(V, W)) = 1$ *if V and W are isomorphic, and* $\dim(\mathrm{Hom}_{\mathfrak{A}}(V, W)) = 0$ *otherwise.*

5.2. **Semisimple Lie algebras and q-deformations.** Special kinds of examples of associative algebras are the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of Lie algebras[20] $\mathfrak{g}$. They consist of formal linear combinations of words with letters in $\mathfrak{g}$, together with the relations $xy - yx - [x, y] = 0$. The algebra $\mathcal{U}(\mathfrak{g})$ is uniquely determined by a universality property, containing $\mathfrak{g}$ as a generating subset.

[19] Also some lattice models (the discrete Gaussian free field and the Ising model) carry an action of the Virasoro algebra [HJVK13, HJVK16].

[20] A Lie algebra is a vector space $\mathfrak{g}$ equipped with a Lie bracket, that is, a bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the alternativity property $[x, x] = 0$, and the Jacobi identity $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$. These defining properties also imply that $[\cdot,\cdot]$ is anti-symmetric, that is, we have $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

An important tool in this thesis is the q-deformation $\mathcal{U}_q(\mathfrak{sl}_2)$ of the enveloping algebra of the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$, which is ubiquitous in the theory of Lie algebras (any semisimple Lie algebra includes copies of $\mathfrak{sl}_2$ as building blocks), as well as in physical systems with spin. The Lie algebra $\mathfrak{sl}_2$ has a basis e, f, h, with commutation relations $[e,f] = h$, $[h,e] = 2e$, and $[h,f] = -2f$. One can represent $\mathfrak{sl}_2$ as the algebra of traceless two-by-two complex matrices, with $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Analogously, for $q \in \mathbb{C} \setminus \{0, \pm 1\}$, the q-deformed algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by the elements E, F, K (corresponding to $e, f, h \in \mathfrak{sl}_2$, respectively), and the inverse K^{-1} of K, with the q-deformed relations

$$KK^{-1} = 1 = K^{-1}K, \qquad KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{1}{q - q^{-1}} \left(K - K^{-1}\right).$$

Another relevant example for us is the Virasoro algebra $\mathfrak{Vir}$, the infinite-dimensional Lie algebra generated by $(\mathrm{L}_n)_{n \in \mathbb{Z}}$ and a central element C, with the commutation relations

$$[\mathrm{L}_n, \mathrm{C}] = 0 \qquad \text{and} \qquad [\mathrm{L}_n, \mathrm{L}_m] = (n-m)\,\mathrm{L}_{n+m} + \frac{1}{12} n(n^2-1)\delta_{n,-m}\,\mathrm{C} \quad \text{for } n, m \in \mathbb{Z}. \tag{5.1}$$

It is the unique central extension of the Witt algebra by the one-dimensional abelian Lie algebra $\mathbb{C}$, arising in CFT in the quantization of the classical conformal symmetry. We consider the role of $\mathfrak{Vir}$ in CFT in Section 6. The central part of $\mathfrak{Vir}$ represents a conformal anomaly, giving rise to a projective representation of the Witt algebra, see e.g. [Sch08, Sections 3 – 4] for details. Note that also the Lie algebra $\mathfrak{Vir}$ contains $\mathfrak{sl}_2$ as a subalgebra: $\mathrm{L}_1, \mathrm{L}_0, \mathrm{L}_{-1}$ correspond with $-f$, $\frac{1}{2}h$, $e \in \mathfrak{sl}_2$, respectively.

5.3. **Highest weight representations and complete reducibility.** The representation theory of semisimple Lie algebras $\mathfrak{g}$ is well studied, see e.g. [Hum72, FH04]. Importantly, any finite-dimensional representation of $\mathfrak{g}$ is completely reducible, and the finite-dimensional irreducibles are generated by highest weight vectors. Recall also that representations of $\mathfrak{g}$ correspond one-to-one[21] with those of $\mathcal{U}(\mathfrak{g})$.

For instance, the irreducible representations of the semisimple Lie algebra $\mathfrak{sl}_2$ are highest weight representations with integer weights. More precisely, for any $\lambda \in \mathbb{Z}_{\geq 0}$, there exists an irreducible representation V_d of $\mathfrak{sl}_2$ of dimension $d = \lambda + 1$, generated by a highest weight vector $v_0 \in \mathsf{V}_d$ of weight λ, that is, a vector satisfying $e.v_0 = 0$ and $h.v_0 = \lambda v_0$. Conversely, $\mathfrak{sl}_2$ has no other finite-dimensional irreducibles. The representation V_d has a basis $v_j = f^j v_0$, for $j = 0, 1, \ldots, d-1$, with the generators e, f, h acting by

$$h.v_j = (d-1-2j)v_j, \qquad e.v_j = j(d-j)v_{j-1}, \qquad \text{and} \qquad f.v_j = v_{j+1}.$$

In the sense of quantum mechanics, $\frac{1}{2}h$ is a component of the spin angular momentum operator, and e and f act as raising and lowering operators. The highest weight vector v_0 corresponds to a highest spin state (and v_{d-1} the lowest). The d-dimensional irreducibles V_d of $\mathfrak{sl}_2$ are "spin $\frac{d-1}{2}$-representations".

Semisimplicity of $\mathfrak{sl}_2$ has an important consequence for us: any finite-dimensional representation of $\mathfrak{sl}_2$ is completely reducible. In particular, finite tensor products[22] of the irreducible representations V_d decompose according to the famous Clebsch-Gordan formula

$$\mathsf{V}_d \otimes \mathsf{V}_{d'} \;\cong\; \mathsf{V}_{d+d'-1} \oplus \mathsf{V}_{d+d'-3} \oplus \cdots \oplus \mathsf{V}_{|d-d'|+3} \oplus \mathsf{V}_{|d-d'|+1}. \tag{5.2}$$

When the deformation parameter $q \in \mathbb{C} \setminus \{0, \pm 1\}$ is not a root of unity, semisimplicity of $\mathfrak{sl}_2$ is inherited to the representation theory of the q-deformed algebra $\mathcal{U}_q(\mathfrak{sl}_2)$. It has irreducible representations M_d of all dimensions, q-deformed analogues of the representations V_d of $\mathfrak{sl}_2$, and a formula similar to (5.2) holds. They have a basis $e_j = F^j e_0$, for $j = 0, 1, \ldots, d-1$, with the generators E, F, K acting by

$$K.e_j = q^{d-1-2j}\, e_j, \qquad E.e_j = \frac{(q^j - q^{-j})(q^{d-j} - q^{j-d})}{(q - q^{-1})^2}\, e_{j-1}, \qquad \text{and} \qquad F.e_j = e_{j+1}.$$

[21]With abuse of terminology, we do not distinguish representations of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$, although the former are homomorphisms $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras, and the latter homomorphisms $\rho\colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ of associative algebras.

[22]In Section 5.5, we explain how one can define tensor products of representations of Lie algebras $\mathfrak{g}$.

5.4. **Representation theory of the Virasoro algebra.** Also the infinite-dimensional Lie algebra $\mathfrak{Vir}$ has highest weight representations $V = \mathcal{U}(\mathfrak{Vir}).v_0$, each generated by a highest weight vector v_0 of weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$, that is, a vector $v_0 \in V$ satisfying $\mathrm{L}_0.v_0 = h\, v_0$, $\mathrm{L}_n.v_0 = 0$ for $n \geq 1$, and $\mathrm{C}.v_0 = c\, v_0$. The operator L_0 now plays the role of the energy operator, and L_n with $n \geq 1$ are lowering operators — v_0 is the lowest energy state and L_{-n} with $n \geq 1$ give excited states.

For any pair (c, h), there exists a unique Verma module $\mathrm{M}_{c,h} = \mathcal{U}(\mathfrak{Vir})/\mathcal{I}_{c,h}$ (up to isomorphism), where $\mathcal{I}_{c,h}$ is the left ideal generated by the elements $\mathrm{L}_0 - h\,1$, $\mathrm{C} - c\,1$, and L_n for $n \geq 1$. The Verma module $\mathrm{M}_{c,h}$ is a highest weight representation, generated by a highest weight vector $v_{c,h}$ of weight h and central charge c (given by the equivalence class of the unit 1). It has a Poincaré-Birkhoff-Witt type basis $\{\mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k}.v_{c,h} \mid n_1 \geq \cdots \geq n_k > 0,\ k \in \mathbb{Z}_{\geq 0}\}$. In fact, $\mathrm{M}_{c,h}$ are universal in the sense that if V is any representation of $\mathfrak{Vir}$ containing a highest weight vector v_0 of weight h and central charge c, then there exists a canonical homomorphism $\varphi\colon \mathrm{M}_{c,h} \to V$ such that $\varphi(v_{c,h}) = v_0$, see e.g. [IK11].

Quotients of Verma modules by maximal proper submodules produce irreducible representations of the Virasoro algebra. The L_0-eigenvalue of a basis vector $v = \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k}.v_{c,h} \in \mathrm{M}_{c,h}$ can be calculated using the commutation relations (5.1): we have $\mathrm{L}_0.v = (h + \sum_{i=1}^k n_i)v = (h + \ell)v$. The number $\ell := \sum_{i=1}^k n_i$ is called the level of the vector v. Submodules of Verma modules were classified by Feĭgin and Fuchs [FF82, FF84, FF90], who showed that every non-trivial submodule of a Verma module $\mathrm{M}_{c,h}$ is generated by some singular vectors — a vector $v \in \mathrm{M}_{c,h} \setminus \{0\}$ is said to be singular at level $\ell \in \mathbb{Z}_{>0}$ if it has the properties

$$\mathrm{L}_0.v = (h + \ell)v \qquad \text{and} \qquad \mathrm{L}_n.v = 0 \quad \text{for } n \geq 1. \tag{5.3}$$

Furthermore, Feĭgin and Fuchs found a characterization for the existence of singular vectors in terms of the Kac determinant [Kac79, FF82, FF84]: the Verma module $\mathrm{M}_{c,h}$ contains a singular vector if and only if there exist $r, s \in \mathbb{Z}_{>0}$ and $t \in \mathbb{C} \setminus \{0\}$ such that

$$h = h_{r,s}(t) := \frac{(r^2-1)}{4}\,t + \frac{(s^2-1)}{4}\,t^{-1} + \frac{(1-rs)}{2} \qquad \text{and} \qquad c = c(t) = 13 - 6(t + t^{-1}). \tag{5.4}$$

In this case, the smallest $\ell = rs$ is the lowest level at which a singular vector occurs in $\mathrm{M}_{c,h}$. Otherwise, $\mathrm{M}_{c,h}$ is irreducible. The weights $h = h_{r,s}$ are the roots of the Kac determinant [Kac79, Kac80], often called Kac conformal weights. For instance, one can check that $\mathrm{L}_{-1}.v_{c,h}$ is a singular vector at level one if and only if $h = h_{1,1} = 0$. As a more involved example, let $a \in \mathbb{C}$ and consider the ansatz

$$v = (\mathrm{L}_{-2} + a\mathrm{L}_{-1}^2).v_{c,h} \tag{5.5}$$

for a singular vector at level two. The definition (5.3) implies that we must have $a = -\frac{3}{2(2h+1)}$ and $h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$, which equals $h_{1,2}$ or $h_{2,1}$ depending on the choice of sign.

In general, explicit expressions for singular vectors are hard to find — one has to construct a suitable polynomial P so that the vector $v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots).v_{c,h}$ is singular. Remarkably, in the case when either $r = 1$ or $s = 1$, Louis Benoit and Yvan Saint-Aubin managed to find a family of such vectors in [BSA88]. For $r = 1$ and $s \in \mathbb{Z}_{>0}$, the singular vector at level $\ell = s$ has the formula

$$\sum_{k=1}^{s} \sum_{\substack{\{n_1,\ldots,n_k\} \in \mathbb{Z}_{>0}^k, \\ n_1+\ldots+n_k = s}} \frac{(-t)^{k-s}\,(s-1)!^2}{\prod_{j=1}^{k-1}(\sum_{l=1}^{j} n_l)(\sum_{l=j+1}^{k} n_l)} \times \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k}.v_{c,h_{1,s}}. \tag{5.6}$$

The case $s = 1$ and $r \in \mathbb{Z}_{>0}$ is obtained by taking $t \mapsto t^{-1}$. In [BFIZ91], Bauer, Di Francesco, Itzykson, and Zuber found the general singular vectors, for $r, s \in \mathbb{Z}_{>0}$, using a fusion procedure. The formulas for these expressions, however, are not explicit.

Remark. Singular vectors give rise to degeneracies in CFT, resulting in PDEs for correlation functions of the CFT — see Section 6.4 for details. Such PDEs are also present in the theory of SLE_κ, with the parameter chosen by $t = \kappa/4$, in which case we have $c = \frac{1}{2\kappa}(6-\kappa)(3\kappa-8)$. For example, the vector (5.5) gives rise to the PDEs (4.5) and (8.5), when translation invariance of the solutions is assumed.

5.5. **Building more representations: counit and coproduct.** The field $\mathbb{C}$ is a subspace of any associative $\mathbb{C}$-algebra $\mathfrak{A}$ with unit, via the embedding $\lambda \mapsto \lambda 1_{\mathfrak{A}}$. Conversely, if $\mathfrak{A}$ has a counit, that is, a homomorphism $\epsilon\colon \mathfrak{A} \to \mathbb{C}$ of associative algebras, then one can canonically define a representation of $\mathfrak{A}$ on the one-dimensional vector space $\mathbb{C}$, called a trivial representation. For instance, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ always has the trivial representation $\epsilon\colon \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ defined by setting $\epsilon(x) = 0$ for all $x \in \mathfrak{g}$, and extending by the algebra homomorphism property.

If V and W are two representations of $\mathfrak{A}$, one can define an action of $\mathfrak{A}$ on the direct sum $V \oplus W$ by $a.(v, w) := (a.v, a.w)$. Also, the tensor product algebra $\mathfrak{A}\otimes\mathfrak{A}$ acts naturally on $V\otimes W$ by $(a\otimes b).(v\otimes w) = a.v \otimes b.w$. However, $\mathfrak{A}$ itself does not always admit an action on the tensor product $V \otimes W$. In general, what is needed for defining a representation $\rho\colon \mathfrak{A} \to \mathrm{End}(V \otimes W)$ is a prescription of how to "share out" elements of $\mathfrak{A}$. This is provided by a coproduct map, i.e., a homomorphism $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ of algebras. Indeed, the assignment $\rho(a)(v \otimes w) := \Delta(a).(v \otimes w)$ then defines a representation of $\mathfrak{A}$ on $V \otimes W$.

In the case of Lie algebras, the coproduct on $\mathcal{U}(\mathfrak{g})$ is uniquely determined by setting $\Delta(x) = x\otimes 1+1\otimes x$ for $x \in \mathfrak{g}$, and extending by the algebra homomorphism property. For q-deformations $\mathcal{U}_q(\mathfrak{g})$, the coproduct is less symmetric, which yields non-trivial braiding properties of the tensor product representations, as discussed in Section 5.8. For instance, the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ has the coproduct

$$\Delta(E) \,=\, E \otimes K + 1 \otimes E, \qquad \Delta(K) \,=\, K \otimes K, \qquad \Delta(F) \,=\, F \otimes 1 + K^{-1} \otimes F.$$

5.6. **Dual representations.** As a warm-up, consider a representation V of a finite group G. The dual representation structure on $V^* = \mathrm{Hom}(V, \mathbb{C})$ is defined by $(g.\varphi)(v) := \varphi(g^{-1}.v)$ for $g \in G$, $v \in V$, and $\varphi \in V^*$. The map $g \mapsto g^{-1}$ is an anti-homomorphism of groups, i.e., it reverses the order of the product. This means that the map $\rho(g)v = v.g := g^{-1}.v$ defines a right action of G on V.

Using the same idea, for an algebra $\mathfrak{A}$, one can define a left action $(a.\psi)(b) := \psi(ba)$ on its own dual $\mathfrak{A}^*$, for $a, b \in \mathfrak{A}$ and $\psi \in \mathfrak{A}^*$, applying the right action of $\mathfrak{A}$ on itself, by multiplication from the right.

However, for a general representation V of $\mathfrak{A}$, to define a left action of $\mathfrak{A}$ on the dual V^*, we first need to make sense of how to build a right action of $\mathfrak{A}$ on V from the left action. This is established with an anti-homomorphism $\gamma\colon \mathfrak{A} \to \mathfrak{A}$ of algebras, i.e., a linear map such that $\gamma(ab) = \gamma(b)\gamma(a)$ and $\gamma(1_{\mathfrak{A}}) = 1_{\mathfrak{A}}$. If there exists such an anti-homomorphism γ, then, for any representation V of $\mathfrak{A}$, the assignment $(a.\psi)(v) := \psi(\gamma(a).v)$ for $a \in \mathfrak{A}$, $v \in V$, and $\psi \in V^*$, defines a representation of $\mathfrak{A}$ on V^*.

More generally, consider the space $\mathrm{Hom}(V, W)$ of linear maps between two representations V and W of $\mathfrak{A}$. Recall that, for a linear map $\psi\colon V \to W$, the transpose map $\psi^T\colon W^* \to V^*$ is defined by $\psi^T(\varphi)(v) := \varphi(\psi(v))$ for $\varphi \in W^*$ and $v \in V$. We already used the transpose map above, with $W = \mathbb{C}$:

$$(a.\psi)(v) \,:=\, \psi(\gamma(a).v) \,=\, \psi\big(\rho_V(\gamma(a))(v)\big) \,=\, \big(\rho_V(\gamma(a))\big)^T(\psi)(v)$$

for a representation $\rho_V : \mathfrak{A} \to \mathrm{End}(V)$. To define an action of $\mathfrak{A}$ on $\mathrm{Hom}(V, W)$, we generalize this as follows. Suppose that $\mathfrak{A}$ has a coproduct $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$. Write $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$, emphasizing that such an expression means any choice of a sum of simple tensors $a_{(1)} \otimes a_{(2)}$ formed by suitable elements $a_{(1)}, a_{(2)} \in \mathfrak{A}$. Then, because Δ is a homomorphism, and $\gamma\colon \mathfrak{A} \to \mathfrak{A}$ an anti-homomorphism, a direct calculation shows that the following assignment defines a representation of $\mathfrak{A}$ on $\mathrm{Hom}(V, W)$:

$$(a.T)(v) \,:=\, \sum_{(a)} a_{(1)}.\big(T\big(\gamma(a_{(2)}).v\big)\big) \qquad \text{for } \ a \in \mathfrak{A}, \ v \in V, \text{ and } \ T \in \mathrm{Hom}(V, W).$$

5.7. **Hopf algebras.** Let $\mathfrak{A}$ be a $\mathbb{C}$-vector space equipped with the following (multi)linear maps:

- the product $\mu\colon \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, defined by $\mu(a \otimes b) = ab$,
- the unit $\iota\colon \mathbb{C} \to \mathfrak{A}$, defined by $\iota(\lambda) = \lambda 1_{\mathfrak{A}}$,
- the coproduct $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$,
- the counit $\epsilon\colon \mathfrak{A} \to \mathbb{C}$,
- and the antipode $\gamma\colon \mathfrak{A} \to \mathfrak{A}$.

We say that $\mathfrak{A}$ is a Hopf algebra if the following axioms hold.

(A1) **The unitality axiom**: we have $\mu \circ (\iota \otimes \mathrm{id}_{\mathfrak{A}}) = \mathrm{id}_{\mathfrak{A}} = \mu \circ (\mathrm{id}_{\mathfrak{A}} \otimes \iota)$, that is,

$$1_{\mathfrak{A}}\, a = a = a\, 1_{\mathfrak{A}} \qquad \text{for all } \; a \in \mathfrak{A}.$$

(A2) **The associativity axiom**: we have $\mu \circ (\mu \otimes \mathrm{id}_{\mathfrak{A}}) = \mu \circ (\mathrm{id}_{\mathfrak{A}} \otimes \mu)$, that is,

$$(ab)c = a(bc) \qquad \text{for all } \; a, b, c, \in \mathfrak{A}.$$

(C1) **The counitality axiom**: we have $(\epsilon \otimes \mathrm{id}_{\mathfrak{A}}) \circ \Delta = \mathrm{id}_{\mathfrak{A}} = (\mathrm{id}_{\mathfrak{A}} \otimes \epsilon) \circ \Delta$, that is,

$$\sum_{(a)} \epsilon(a_{(1)})\, a_{(2)} = a = \sum_{(a)} a_{(1)}\, \epsilon(a_{(2)}) \qquad \text{for all } \; a \in \mathfrak{A},$$

where we canonically identify $\mathbb{C} \otimes \mathfrak{A} \cong \mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{C}$.

(C2) **The coassociativity axiom**: we have $(\Delta \otimes \mathrm{id}_{\mathfrak{A}}) \circ \Delta = (\mathrm{id}_{\mathfrak{A}} \otimes \Delta) \circ \Delta$, that is,

$$\sum_{(a)} \Delta(a_{(1)}) \otimes a_{(2)} = \sum_{(a)} a_{(1)} \otimes \Delta(a_{(2)}) \qquad \text{for all } \; a \in \mathfrak{A}.$$

(H) **The antipode axiom**: we have $\mu \circ (\gamma \otimes \mathrm{id}_{\mathfrak{A}}) \circ \Delta = \iota \circ \epsilon = \mu \circ (\mathrm{id}_{\mathfrak{A}} \otimes \gamma) \circ \Delta$, that is,

$$\sum_{(a)} \gamma(a_{(1)})\, a_{(2)} = \iota\big(\epsilon(a)\big) = \sum_{(a)} a_{(1)}\, \gamma(a_{(2)}) \qquad \text{for all } \; a \in \mathfrak{A}.$$

($\star$) **Compatibility**: the maps $\epsilon \colon \mathfrak{A} \to \mathbb{C}$ and $\Delta \colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ are homomorphisms of algebras.

Vector spaces $\mathfrak{A}$ satisfying the axioms (A1), (A2) are just associative algebras with unit $1_{\mathfrak{A}}$, and vector spaces satisfying the axioms (C1), (C2) — obtained by "reversing the arrows" — are called co-algebras. In the finite-dimensional case, they are dual to each other in the sense that the dual space $\mathfrak{A}^* = \mathrm{Hom}(\mathfrak{A}, \mathbb{C})$ of an algebra $\mathfrak{A}$ has a natural co-algebra structure, and vice versa, see e.g. [Kas95].

The axioms (C1) and (C2) imply that the tensor product representations of $\mathfrak{A}$ have useful properties. The counitality axiom (C1) provides a neutral element for the tensor product operation: for any representation V of $\mathfrak{A}$, we may canonically identify $\mathbb{C} \otimes V \;\cong\; V \;\cong\; V \otimes \mathbb{C}$ as representations of $\mathfrak{A}$.

One would also like say that, when "sharing out" an element of $\mathfrak{A}$ twice, it does not matter which of the two tensor factors obtained in the first sharing is shared in the second step. This is accomplished by the coassociativity axiom (C2). We can identify all triple tensor products of $\mathfrak{A}$-modules,

$$(V \otimes W) \otimes U \;\cong\; V \otimes (W \otimes U) \;\cong\; V \otimes W \otimes U,$$

as representations of $\mathfrak{A}$, and omit the parenthesis. The action of $\mathfrak{A}$ on the triple tensor product reads

$$a.(v \otimes w \otimes u) \;=\; \Delta^{(3)}(a).(v \otimes w \otimes u) \;:=\; (\Delta \otimes \mathrm{id}_{\mathfrak{A}})(\Delta(a)).(v \otimes w \otimes u) \;=\; (\mathrm{id}_{\mathfrak{A}} \otimes \Delta)(\Delta(a)).(v \otimes w \otimes u).$$

The map γ satisfying (H) is called an antipode. It follows from the defining properties of the various maps that the antipode γ is an anti-homomorphism of algebras, allowing one to define dual representations and representations on Hom-spaces. Moreover, one can show that if γ satisfying (H) exists, then it is unique, see e.g. [EGNO15]. Interestingly, the axiom (H) guarantees the property (5.7) given below, which provides a characterization of semisimplicity of Hopf algebras, see [Kyt11, Proposition 3.53]. It can be used e.g. to prove that, when the deformation parameter q is not a root of unity, the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is a semisimple Hopf algebra, see [Kyt11, Corollary 4.29].

Lemma. [Kyt11, Lemma 3.55] *Let $\mathfrak{A}$ be a Hopf algebra. Denote by*

$$\mathrm{Hom}_{\mathfrak{A}}(V, W) := \{T \in \mathrm{Hom}(V, W) \mid a.T(v) = T(a.v) \;\; \textit{for all} \;\; a \in \mathfrak{A},\, v \in V\}$$

the space of $\mathfrak{A}$-intertwiners $V \to W$, and define the trivial part of the $\mathfrak{A}$-module $\mathrm{Hom}(V, W)$ *by*

$$\mathrm{Hom}(V, W)^{\mathfrak{A}} := \{T \in \mathrm{Hom}(V, W) \mid a.T = \epsilon(a) T \;\; \textit{for all} \;\; a \in \mathfrak{A}\}\,.$$

Then, we have

$$\mathrm{Hom}_{\mathfrak{A}}(V, W) = \mathrm{Hom}(V, W)^{\mathfrak{A}}. \tag{5.7}$$

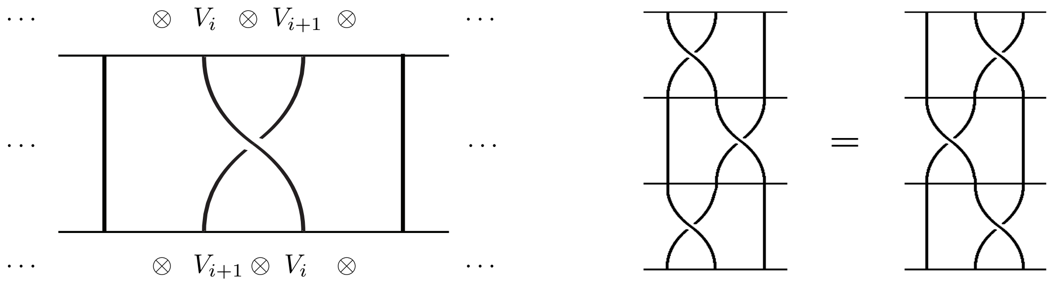


FIGURE 5.1. The left figure depicts braiding of two tensor components in a tensor product representation $V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n$ of a Hopf algebra $\mathfrak{A}$ by the braid group generator $\sigma_i \in \mathfrak{Br}_n$. In the right figure, the Yang-Baxter equation (also known as star-triangle relation) is illustrated in terms of the relations of the braid group.

Remark. One sometimes also requires from a Hopf algebra $\mathfrak{A}$ that the antipode S is invertible. Then, representations of $\mathfrak{A}$ form a tensor category, see [EGNO15]. This is not too much of a complication, as the antipode of a finite-dimensional Hopf algebra is always invertible [Rad76]. Also, braided Hopf algebras, briefly discussed in the next section, always have invertible antipodes, see [KRT97].

5.8. **Braiding of tensor product representations.** One might ask whether the representation structure of $\mathfrak{A}$ on $V \otimes W$ changes when flipping the tensor components. In general, the tensor flip $v \otimes w \mapsto w \otimes v$ is an isomorphism of representations of $\mathfrak{A}$ if $\tau \circ \Delta = \Delta$. If this is not the case, one might hope that the representations $V \otimes W$ and $W \otimes V$ are still isomorphic, with a "twisted" $\mathfrak{A}$-intertwiner $\mathscr{R} \colon V \otimes W \to W \otimes V$, braiding the tensor components. Then, one has to keep track of the entanglement of the braided strands, which leads to representations of the braid group $\mathfrak{Br}_n$, see Figure 5.1. $\mathfrak{Br}_n$ is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, and $\sigma_j \sigma_k = \sigma_k \sigma_j$ for $|j-k| \geq 2$.

In general, suppose that for all finite-dimensional representations V, W of a Hopf algebra $\mathfrak{A}$, there exist invertible maps $\mathscr{R}_{V,W} \in \mathrm{Hom}_{\mathfrak{A}}(V \otimes W, W \otimes V)$ which satisfy the Yang-Baxter equation

$$(\mathscr{R}_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes \mathscr{R}_{U,W}) \circ (\mathscr{R}_{U,V} \otimes \mathrm{id}_W) \; = \; (\mathrm{id}_W \otimes \mathscr{R}_{U,V}) \circ (\mathscr{R}_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \mathscr{R}_{V,W}).$$

Then, the action of these maps on the tensor components of the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$,

$$\sigma_i \; \mapsto \; \mathscr{R}_i \; := \; \mathrm{id}_{V_1} \otimes \cdots \otimes \mathrm{id}_{V_{i-1}} \otimes \mathscr{R}_{V_i, V_{i+1}} \otimes \mathrm{id}_{V_{i+2}} \otimes \cdots \otimes \mathrm{id}_{V_n},$$

defines an "action" of $\mathfrak{Br}_n$ commuting with the action of $\mathfrak{A}$, see also Figure 5.1. This it not quite a representation, as the image space has the i:th and $(i+1)$:st tensor components flipped. Therefore, when braiding general tensor products, it is convenient to consider the normal subgroup $\mathfrak{PBr}_n$ of the braid group $\mathfrak{Br}_n$ called the pure braid group, defined as the kernel of the surjective homomorphism $\sigma_i \mapsto (i, i+1)$ from $\mathfrak{Br}_n$ onto the permutation group $\mathfrak{S}_n$, where $(i, i+1) \in \mathfrak{S}_n$ denotes the transposition. The order of the tensor components remains unchanged under the pure braids, and therefore, $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is a representation of $\mathfrak{PBr}_n$.

For instance, so called universal R-matrices of braided Hopf algebras $\mathfrak{A}$ define representations of the braid group on tensor products of n representations of $\mathfrak{A}$, see e.g. [KRT97]. The tensor products then become representations of both the algebra $\mathfrak{A}$ and the pure braid group, and the two actions commute.

For the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, one can define an "R-matrix" $\mathscr{R}$ which gives solutions to the Yang-Baxter equation on any finite-dimensional tensor product representation, and thus produce a braided structure on tensor products of representations. In particular, on tensor products of the irreducible representations $\mathsf{M}_d = \mathrm{span}\,\{e_0, e_1, \ldots, e_{d-1}\}$ of $\mathcal{U}_q(\mathfrak{sl}_2)$, the operator explicitly reads as follows:

$$\mathscr{R} = \mathscr{R}_{\mathsf{M}_d, \mathsf{M}_{d'}} \; : \; \mathsf{M}_d \otimes \mathsf{M}_{d'} \; \to \; \mathsf{M}_{d'} \otimes \mathsf{M}_d,$$

$$
(5.8) \qquad \mathscr{R}(e_l \otimes e_k) \;=\; \sum_{m=0}^{k} r^m_{l,k}(d',d) \times (e_{k-m} \otimes e_{l+m}), \qquad \text{where}
$$
$$
r^m_{l,k}(d',d) \;=\; q^{2(l-\frac{d-1}{2})(\frac{d'-1}{2}-k)-\frac{m(m-1)}{2}} \, (q^{-1}-q)^m \, \frac{[k]![d'-1+m-k]!}{[k-m]![d'-1-k]![m]!},
$$

and $[k]! = \prod_{i=1}^{k} \frac{q^i-q^{-i}}{q-q^{-1}}$, see e.g. [Pel12] or [FW91, Kas95] (beware of the different conventions, though).

6. Partial differential equations from conformal field theory

In this section, we very briefly describe some aspects of conformal field theory (CFT). There exist many textbooks about CFT from different viewpoints, see e.g [DFMS97, Sch08][23]. We will not present the needed background here, but only give some rough ideas motivating the topics of this thesis. We consider CFT algebraically, in terms of the representation theory of the Virasoro algebra. Importantly, the Benoit & Saint-Aubin PDEs, which are a major topic in this thesis, arise from singular vectors in Virasoro representations carried by families of conformal fields in CFT, as explained in Section 6.4.

6.1. Conformal invariance.

The scaling limits of critical lattice models are expected to enjoy conformal invariance. The conformal maps on the complex plane $\mathbb{C}$ form a group of finite dimension, the Möbius group $\mathrm{PSL}(2,\mathbb{C})$, acting as Möbius transformations $\mu(z) = \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{C}$ and $ad-bc=1$. In particular, global conformal invariance only results in finitely many constraints in the physical system. However, Belavin, Polyakov, and Zamolodchikov observed in the 1980s that, in two dimensions, local conformal invariance in fact yields infinitely many independent symmetries [BPZ84a, BPZ84b][24].

In CFT à la Belavin, Polyakov, and Zamolodchikov, one regards the local conformal invariance as an invariance under infinitesimal transformations (vector fields which generate the local conformal mappings). The infinitesimal holomorphic transformations are written as Laurent series, $z \mapsto z + \sum_{n\in\mathbb{Z}} a_n z^n$. They are generated by the vector fields $\ell_n = -z^{n+1}\frac{\partial}{\partial z}$, which constitute a Lie algebra isomorphic to the Witt algebra, with commutation relations $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$.

Now, the Virasoro algebra $\mathfrak{Vir}$, studied in Section 5.2, arises in quantization as the unique central extension of the Witt algebra by the one-dimensional abelian Lie algebra $\mathbb{C}$. In quantized systems, the symmetry groups and algebras often are central extensions of their classical counterparts, see e.g. [Sch08, Sections 3 – 4]. Algebraically, the basic objects in a CFT, the conformal fields, can be regarded as elements in representations of $\mathfrak{Vir}$, where the central element acts as a constant multiple of the identity, $\mathrm{C} = c\mathbb{1}$. The number $c \in \mathbb{C}$ is called the central charge of the CFT (conformal anomaly).

6.2. Correlation functions.

CFT correlation functions of are of specific interest, as they are related to physical observables. They are analytic, multi-valued functions $G\colon \mathfrak{W}_n \to \mathbb{C}$ on the configuration space

$$
(6.1) \qquad \mathfrak{W}_n \;:=\; \left\{ (z_1,\ldots,z_n) \in \mathbb{C}^n \;\middle|\; z_i \neq z_j \;\text{ for }\; i \neq j \right\},
$$

also called n-point functions. The physical correlation functions should be single-valued on (6.1), that is, invariant under analytic continuation around singularities (i.e., monodromy invariant). Note that the fundamental group of $\mathfrak{W}_n$ is the braid group $\mathfrak{Br}_n$, and the monodromy of the correlation functions thus defines an action of $\mathfrak{Br}_n$. The monodromy properties of certain correlation functions are considered in the series of articles [D, E, F, G], see also Section 8.6.

Physicists speak of correlation functions as vacuum expectation values of local fields $\Phi^{(i)}(z_i)$ and denote

$$
G(z_1,\ldots,z_n) \;=\; \langle \Phi^{(1)}(z_1) \cdots \Phi^{(n)}(z_n) \rangle.
$$

In Section 7, we will make sense of such vacuum expectations in terms of vertex operators.

[23] The former is a physics introduction and the latter a mathematical one.

[24] Local conformal transformations are defined as smooth mappings of maximal rank that scale the metric by a non-zero factor. By the Cauchy-Riemann equations, on $\mathbb{C}$ these are just the locally invertible holomorphic and anti-holomorphic maps — see e.g. [Sch08, Sections 1 – 2] for more details.

One often considers correlation functions which are covariant under global conformal transformations. In a CFT on the full plane $\mathbb{C}$, this is a specific transformation property under all Möbius transformations μ,

$$G(z_1,\ldots,z_n) \;=\; \prod_{i=1}^{n} |\mu'(z_i)|^{h_i} \times G\big(\mu(z_1),\ldots,\mu(z_n)\big), \tag{6.2}$$

with some conformal weights $h_i \in \mathbb{R}$ of the fields $\Phi^{(i)}$. Of specific interest to us is CFT in the domain $\Lambda = \mathbb{H}$ with boundary $\partial\mathbb{H} = \mathbb{R}$, where the global conformal transformations are also Möbius maps, $\mu \in \mathrm{PSL}(2,\mathbb{R})$. For example, the multiple SLE_κ partition functions $\mathcal{Z}(x_1,\ldots,x_{2N})$ satisfy the covariance property (6.2), with $h_i = h_{1,2} = \frac{6-\kappa}{2\kappa}$ for all i, see Equation (8.6) in Section 8.

6.3. **Primary fields.** Fields whose correlation functions have a covariance property under local conformal transformations as well, have a special role. These are called primary fields. They are defined as fields $\Psi_h(z)$ satisfying the following Virasoro intertwining relation, with some conformal weight $h \in \mathbb{R}$,

$$[\mathrm{L}_n, \Psi_h(z)] \;=\; \Big(z^{n+1}\frac{\partial}{\partial z} \;+\; h(n+1)z^n\Big)\,\Psi_h(z) \qquad \text{for all } \mathrm{L}_n,\; n\in\mathbb{Z}. \tag{6.3}$$

To motivate this, suppose for a moment that there would exist a conformal “Lie group” corresponding to the Lie algebra $\mathfrak{Vir}$ (this is not true, though, see e.g. [Sch08, Section 5] !!!), so that the group elements would be of the form $e^{t\mathrm{L}_n}$ for small t. Then the transformation $\mu(z) = z + tz^{n+1} + \mathcal{O}(t^2)$, implemented by the Virasoro generator L_n, should give for any primary field $\Psi_h(z)$ the transformation rule

$$e^{t\mathrm{L}_n}\Psi_h(z)e^{-t\mathrm{L}_n} \;=\; \big(\mu'(z)\big)^h\,\Psi_h(\mu(z)) \;\approx\; (1+t(n+1)z^n)^h\Psi_h(z+tz^{n+1}).$$

Taking the differential $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}$ then gives (6.3), by the Baker-Campbell-Hausdorff formula, see e.g [Ros02].

6.4. **Partial differential equations from Virasoro singular vectors.** In CFT, it is postulated that there exists a special field, the energy-momentum tensor $\mathrm{T}(z)$, which implements local deformations of the geometry — see e.g. [Sch08, Section 9] and [DFMS97, Section 6.6] for details[25]. For a product of primary fields $\Psi_{h_i}(z_i)$ of conformal weights h_i, it has the property

$$\big\langle \mathrm{T}(\zeta)\Psi_{h_1}(z_1)\cdots\Psi_{h_n}(z_n)\big\rangle \;=\; \sum_{i=1}^{n}\bigg(\frac{h_i}{(\zeta-z_i)^2} \;+\; \frac{1}{(\zeta-z_i)}\frac{\partial}{\partial z_i}\bigg)\big\langle\Psi_{h_1}(z_1)\cdots\Psi_{h_n}(z_n)\big\rangle. \tag{6.4}$$

The primary field $\Psi_h(z)$ generates a highest weight representation $\mathsf{W}_{c,h}$ of the Virasoro algebra of weight h and central charge c. The central charge is determined by the energy-momentum tensor $\mathrm{T}(z)$ of the theory[26]. In general, $\mathsf{W}_{c,h}$ is a quotient of the Verma module $\mathrm{M}_{c,h}$. In physics, it is called the conformal family of $\Psi_h(z)$, consisting of linear combinations of the descendants $\mathrm{L}_{-n_1}\cdots\mathrm{L}_{-n_k}.\Psi_h(z)$ of $\Psi_h(z)$. The correlation functions of the descendants are determined by the correlation functions of $\Psi_h(z)$ using linear differential operators which arise from the generators L_{-k} of the Virasoro algebra. Namely, for any primary fields $\Psi_{h_i}(z_i)$, and $\Psi_h(z)$, we have [DFMS97, Section 6.6]

$$\big\langle\Psi_{h_1}(z_1)\cdots\Psi_{h_n}(z_n)\;\mathrm{L}_{-k}\Psi_h(z)\big\rangle \;=\; \mathcal{L}_{-k}^{(z)}\,\langle\Psi_{h_1}(z_1)\cdots\Psi_{h_n}(z_n)\Psi_h(z)\rangle, \qquad \text{where}$$

$$\mathcal{L}_{-k}^{(z)} \;:=\; \sum_{i=1}^{n}\bigg(\frac{(k-1)h_i}{(z_i-z)^k} \;-\; \frac{1}{(z_i-z)^{k-1}}\frac{\partial}{\partial z_i}\bigg) \qquad \text{for } \; k\geq 1. \tag{6.5}$$

Consider the Virasoro representation $\mathsf{W}_{c,h}$ generated by a primary field $\Psi_h(z)$. It is a quotient of a Verma module, $\mathsf{W}_{c,h} = \mathrm{M}_{c,h}/\mathcal{I}$. Suppose that $h = h_{r,s}$, as in (5.4). Then, $\mathrm{M}_{c,h_{r,s}}$ contains a singular vector $v = P(\mathrm{L}_{-1},\mathrm{L}_{-2},\ldots).v_{c,h_{r,s}}$ at level rs, where P is some polynomial. Now, if the singular vector v is contained in $\mathcal{I}$, then the descendant field $\Phi(z) = P(\mathrm{L}_{-1},\mathrm{L}_{-2},\ldots).\Psi_{h_{r,s}}(z)$ corresponding to v generates

[25] The energy-momentum tensor can be regarded e.g. as a formal distribution $\mathrm{T}(z) = \sum_{n\in\mathbb{Z}}\mathrm{L}_n z^{-n-2}$, where the Fourier coefficients L_n form a representation of the Virasoro algebra, see [Sch08, Section 10].

[26] The central charge c of a conformal field theory with stress-energy tensor $\mathrm{T}(z)$ appears in the transformation rule of T under Möbius maps μ, $\mathrm{T}(z)\mapsto(\mu'(z))^2\mathrm{T}(\mu(z)) + \frac{c}{12}S\mu(z)$, where $S\mu(z) = \frac{\mu'''(z)}{\mu'(z)} - \frac{3}{2}\left(\frac{\mu''(z)}{\mu'(z)}\right)^2$ is the Schwarzian derivative. Note that T is not a primary field.

fields which belong to the equivalence class of $\mathcal{I}$, null fields. In this case, we say that $\Psi_h(z)$ has a degeneracy at level rs. In particular, correlation functions containing $\Psi_h(z)$ thus satisfy PDEs

$$0 \;=\; \langle \Psi_{h_1}(z_1) \cdots \Psi_{h_n}(z_n) \Phi(z) \rangle \;=\; P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots)\, \langle \Psi_{h_1}(z_1) \cdots \Psi_{h_n}(z_n) \Psi_{h_{r,s}}(z) \rangle$$

involving the differential operators (6.5). An example of such a PDE is the second order equation generated by the singular vector (5.5) at level two, associated to the primary field $\Psi_{h_{1,2}}(z) = \Psi_{1,2}(z)$ of Kac conformal weight $h_{1,2}$. Remarkably, this PDE gives rise to equations relevant in the theory of SLE_κ.

- Choosing the primaries $\Psi_{h_{1,3}}(z_i) = \Psi_{1,3}(z_i)$ of Kac weight $h_i = h_{1,3} = \frac{8-\kappa}{\kappa}$ for all i, the above PDE, combined with translation invariance of the correlation function $\langle \Psi_{h_1}(z_1) \cdots \Psi_{h_n}(z_n) \Phi(z) \rangle$, results in the PDE (4.5) for the SLE_κ boundary Green's functions $\zeta(z; z_1, \ldots, z_n)$.
- Similarly, choosing the primaries $\Psi_{h_{1,2}}(z) = \Psi_{1,2}(z_i)$ of Kac weight $h_i = h_{1,2} = \frac{6-\kappa}{2\kappa}$ for all i, we obtain the PDEs (8.5) related to the multiple SLE_κ partition functions $\mathcal{Z}(x_1, \ldots, x_{2N})$, with the variables $\{x_1, \ldots, x_{2N}\} = \{z_1, \ldots, z_n, z\}$ (the correlation function vanishes unless $n+1$ is even).

It can be argued [BB03a] that the starting points $\xi \in \partial\Lambda$ of SLE_κ curves in a domain Λ correspond with insertions of the boundary changing operators[27] $\Psi_{1,2}(\xi)$ to the boundary points ξ of the domain, in a CFT of central charge $c = \frac{1}{2\kappa}(6-\kappa)(3\kappa-8)$. These operators have the degeneracy (5.5) at level two[28]. Furthermore, one can modify the boundary conditions by inserting other fields to the boundary. In particular, insertions of $\Psi_{1,3}(y_i)$ should correspond with conditioning the curve to hit neighborhoods of the boundary points $y_i \in \partial\Lambda$, in the spirit of Section 3.9. Therefore, one might expect that, in addition to the second order PDE (4.5), the SLE_κ boundary Green's functions $\zeta(z; y_1, \ldots, y_n)$ should satisfy the third order PDEs associated to the degeneracies of $\Psi_{1,3}(y_i)$ at level three.

In this thesis, we consider the PDEs which arise from the explicit expressions (5.6) for singular vectors found in [BSA88]. For a singular vector at level s in the Verma module $\mathrm{M}_{c,h_{1,s}}$ generated by the primary field $\Psi_{h_{1,s}}(z) = \Psi_{1,s}(z)$ of Kac weight $h_{1,s}$, these Benoit & Saint-Aubin PDEs are of the form

$$\mathcal{D}_s^{(z)} \langle \Psi^{(1)}(z_1) \cdots \Psi^{(n)}(z_n) \Psi_{1,s}(z) \rangle \;=\; 0, \qquad \text{where} \tag{6.6}$$

$$\mathcal{D}_s^{(z)} \;:=\; \sum_{k=1}^{s} \sum_{\substack{n_j \in \mathbb{Z}_{>0}, \\ n_1 + \ldots + n_k = s}} \frac{(-t)^{k-s}\,(s-1)!^2}{\prod_{j=1}^{k-1} (\sum_{l=1}^{j} n_l)(\sum_{l=j+1}^{k} n_l)} \times \mathcal{L}_{-n_1}^{(z)} \cdots \mathcal{L}_{-n_k}^{(z)}, \qquad \text{and} \;\; t = \kappa/4. \tag{6.7}$$

In the article [A], we construct solutions to these PDEs and study their asymptotic properties. The solution space is further studied in our articles [D, E]. Also in the articles [B, C], we consider solutions to these PDEs, at levels two and three, in applications to the LERW, UST, and the theory of SLEs. As a further application of the methods developed in the first article [A], we construct in [G] the unique solution to the PDEs of type (6.6) which is monodromy invariant, i.e., single-valued on the configuration space (6.1). The contributions of this thesis are discussed in more detail in Section 8.

6.5. **Fusion.** In CFT, it is postulated that the fields constitute an operator algebra $\mathfrak{A}$ with an associative product, the operator product expansion (OPE) [BPZ84b, BPZ84a]. Strictly speaking, the OPEs should be understood in terms of asymptotic expansions of the correlation functions. Using vertex operator algebras, the algebra $\mathfrak{A}$ and its OPE obtain a mathematically clean formulation, see e.g. [Kac98, Sch08]. One usually writes the formal operator product in the asymptotic form, omitting the finite terms,

$$\Phi_{h_i}(z_1)\Phi_{h_j}(z_2) \;\sim\; \sum_k C_{ijk}\,(z_1 - z_2)^{h_k - h_i - h_j}\,\Phi_{h_k}(z_2) \qquad \text{as } |z_1 - z_2| \to 0,$$

where Φ_{h_ℓ} are fields with conformal weights $h_\ell \in \mathbb{R}$ and $C_{ijk} \in \mathbb{C}$ are called structure constants. In this context, physicists speak of fusion rules that tell which fields Φ_{h_k} are present in the OPE of the two fields Φ_{h_i} and Φ_{h_j}, i.e., which of the structure constants C_{ijk} are non-zero. The fusion rules can be used to motivate the choice of specific asymptotic boundary conditions imposed to single out specific

[27] Boundary changing operators appear in CFT in domains with boundary, see e.g. [Car06].

[28] One can also insert $\Psi_{2,1}(\xi)$ of weight $h_{2,1}$, but these are related to $\Psi_{1,2}(\xi)$ by the duality $\kappa \to 16/\kappa$, see e.g. [Dup06].

solutions to the Benoit & Saint-Aubin PDEs of type (6.6). These ideas are exploited rigorously in the articles [B, D, E, G], see also [Car89, Car92, Wat96, GC05, BBK05, Dub15b, JJK16].

7. Solving PDEs: Coulomb gas formalism

In the 1980s, Feĭgin and Fuchs developed (in an unpublished work) a method for finding solutions of integral form to the PDEs (6.6). This method has been further studied since [DF84, DF85], where Dotsenko and Fateev used a contour deformation method to study the monodromy properties of the solutions. They found formulas for the four-point functions in terms of generalized hypergeometric functions. We next briefly summarize this approach, also known as Coulomb gas formalism — see also [TK86, Fel89, FFK89, FW91, GRAS96, DFMS97]. The idea is to construct primary fields as vertex operators from the exponential of the free bosonic field, and further primaries from them, involving integration over so called screening variables which do not affect the conformal transformation properties of the fields. Vacuum expectations of these operators then produce solutions to the PDEs (6.6).

7.1. **Charged Fock space representations.** Consider the Heisenberg algebra: the algebra of harmonic oscillators generated by $(\mathrm{a}_n)_{n\in\mathbb{Z}}$, with the commutation relations $[\mathrm{a}_n, \mathrm{a}_m] = 2n\delta_{n,-m}$. For any charge $\alpha \in \mathbb{C}$, it has a representation generated by the vector v_α defined by the properties

$$\mathrm{a}_0.v_\alpha \;=\; 2\alpha v_\alpha \qquad \text{and} \qquad \mathrm{a}_n.v_\alpha \;=\; 0 \quad \text{for } \; n \geq 1.$$

This representation is called the bosonic Fock space. Introducing an additional parameter[29] $\alpha_0 \in \mathbb{C}$, the Fock space can be given a representation structure of the Virasoro algebra $\mathfrak{Vir}$, with the central charge $c = 1 - 24\alpha_0^2$. The action of $\mathfrak{Vir}$ is defined by the Sugawara construction, setting

$$\mathrm{L}_n \;:=\; \frac{1}{4}\sum_{j\in\mathbb{Z}} :\mathrm{a}_{n-j}\mathrm{a}_j: \;-\; \alpha_0(n+1)\mathrm{a}_n, \qquad \text{where} \qquad :\mathrm{a}_n\mathrm{a}_m: \;=\; \begin{cases} \mathrm{a}_n\mathrm{a}_m & \text{if } \; n \leq m \\ \mathrm{a}_m\mathrm{a}_n & \text{otherwise.} \end{cases} \tag{7.1}$$

One can check that $(\mathrm{L}_n)_{n\in\mathbb{Z}}$ satisfy the commutation relations (5.1) of $\mathfrak{Vir}$ when acting on the space

$$\mathrm{F}_{\alpha,\alpha_0} \;:=\; \bigoplus_{k=0}^{\infty} \bigoplus_{1\leq n_1\leq\cdots\leq n_k} \mathbb{C}\,\mathrm{a}_{-n_1}\cdots\mathrm{a}_{-n_k}.v_\alpha,$$

called the charged Fock space representation. The operators L_n thus defined have only finitely many non-zero terms when acting on any vector $v \in \mathrm{F}_{\alpha,\alpha_0}$. The vector v_α is a Virasoro highest weight vector of weight $h(\alpha) = \alpha^2 - 2\alpha_0\alpha$. Furthermore, the charged Fock space $\mathrm{F}_{\alpha,\alpha_0}$ is naturally graded by the Virasoro generator L_0, that is, $\mathrm{F}_{\alpha,\alpha_0}$ decomposes as a direct sum of finite-dimensional L_0-eigenspaces, with eigenvalues $h(\alpha) + \ell$, for $\ell \in \mathbb{Z}_{\geq 0}$. These L_0-homogeneous components are denoted by $(\mathrm{F}_{\alpha,\alpha_0})_\ell$, and they have a basis $\{\mathrm{a}_{-n_1}\cdots\mathrm{a}_{-n_k}.v_\alpha \mid n_1 \geq \cdots \geq n_k > 0, \sum_i n_i = \ell, k \in \mathbb{Z}_{\geq 0}\}$.

The highest weight vector v_α is interpreted as a vacuum. To construct expectations of the fields, we need the dual representation, that is, the contragredient module $\mathrm{F}^*_{\alpha,\alpha_0}$. It is defined as the direct sum of the duals $(\mathrm{F}_{\alpha,\alpha_0})^*_\ell$ of the L_0-homogeneous components $(\mathrm{F}_{\alpha,\alpha_0})_\ell$, with the action of $\mathfrak{Vir}$ given by the pairing $\langle \mathrm{L}_n.\omega, v\rangle = \langle \omega, \mathrm{L}_{-n}.v\rangle$ for $\omega \in \mathrm{F}^*_{\alpha,\alpha_0}$. The dual vector $v^*_\alpha \in \mathrm{F}^*_{\alpha,\alpha_0}$, defined by the property $\langle v^*_\alpha, v_\alpha\rangle = 1$, is also a Virasoro highest weight vector of weight $h(\alpha)$. In fact, the contragredient module $\mathrm{F}^*_{\alpha,\alpha_0}$ is isomorphic to the charged Fock space $\mathrm{F}_{2\alpha_0-\alpha,\alpha_0}$ as a representation of $\mathfrak{Vir}$.

[29] In the physics literature, the parameter α_0 is called the background charge.

7.2. **Vertex operators.** The basic objects in the Coulomb gas formalism are vertex operators — intertwiners between charged Fock spaces — which provide primary operators of weight $h(\alpha)$. The vertex operators $\mathrm{V}_\alpha(z)$ are defined, morally, in terms of the normal ordered exponential $: e^{\mathfrak{i}\alpha\varphi(z)} :$ of the free massless bosonic field $\varphi(z) = -\mathfrak{i}(\mathrm{a}_0 \log z + \sum_{k\neq 0} \frac{z^k}{k}\mathrm{a}_{-k})$, see e.g [DFMS97]. As in (7.1), the normal ordering ensures that the infinite sum appearing in the exponential of U^+ below has only finitely many non-zero terms when acting on any vector on the Fock space. The operator U^- below produces an infinite linear combination of basis elements of the Fock space and thus, the vertex operators take values in a completion $\hat{\mathrm{F}}_{\beta+\alpha,\alpha_0}$ of the Fock space, defined as the direct product of the L_0-eigenspaces $(\mathrm{F}_{\alpha,\alpha_0})_\ell$.

For fixed $\beta \in \mathbb{C}$, the vertex operator $\mathrm{V}_\alpha(z)$ is defined by

$$\mathrm{V}_\alpha(z) : \mathrm{F}_{\beta,\alpha_0} \to \hat{\mathrm{F}}_{\beta+\alpha,\alpha_0}, \qquad \mathrm{V}_\alpha(z) := z^{2\alpha\beta}\mathrm{U}^-_\alpha(z)\mathrm{U}^+_\alpha(z)\mathrm{T}_\alpha, \qquad \mathrm{U}^\pm_\alpha(z) := \exp\left(\mp\sum_{k=1}^\infty \frac{\alpha z^{\mp k}}{k}\mathrm{a}_{\pm k}\right),$$

where $\mathrm{T}_\alpha \colon \mathrm{F}_{\beta,\alpha_0} \to \mathrm{F}_{\beta+\alpha,\alpha_0}$ maps $v_\beta \mapsto v_{\beta+\alpha}$, and satisfies $[\mathrm{a}_n, \mathrm{T}_\alpha] = 2\alpha\delta_{n,0}$, and the variable z belongs to the universal covering manifold of $\mathbb{C}\setminus\{0\}$. The vertex operators $\Psi_h(z) = \mathrm{V}_\alpha(z)$ satisfy the Virasoro intertwining relations (6.3) of primary fields, with the conformal weight $h = h(\alpha)$.

It was proved by Tsuchiya and Kanie [TK86] that, for $|z_1| > \cdots > |z_n|$, the composed operators

$$\mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n) := \mathrm{V}_{\alpha_1}(z_1)\circ\cdots\circ\mathrm{V}_{\alpha_n}(z_n) : \mathrm{F}_{\beta,\alpha_0} \to \hat{\mathrm{F}}_{\beta+\alpha_1+\cdots+\alpha_n,\alpha_0}$$

make sense, and one can then analytically continue to obtain well defined multi-valued operators on the configuration space (6.1), with $z_i \neq 0$ for all i. In fact, the Baker-Campbell-Hausdorff formula gives

$$\mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n) = \prod_{k=1}^n z_k^{2\alpha_k\beta} \prod_{1\le i<j\le n}(z_j - z_i)^{2\alpha_i\alpha_j}\, \mathrm{U}^-_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n)\, \mathrm{U}^+_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n)\, \mathrm{T}_{\alpha_\infty},$$

$$\mathrm{U}^\pm_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n) := \exp\left(\mp\sum_{k=1}^\infty \frac{\sum_{i=1}^n \alpha_i z_i^{\mp k}}{k}\mathrm{a}_{\pm k}\right),$$

where $\alpha_\infty = \sum_{i=1}^n \alpha_i$. The intertwining relation (6.3) is summed up, that is, we have

$$[\mathrm{L}_n, \mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n)] = \sum_{i=1}^n \left(z_i^{n+1}\frac{\partial}{\partial z_i} + h(\alpha_i)(n+1)z_i^n\right)\mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n). \tag{7.2}$$

The vacuum expectations of the composed operators are

$$\langle \mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n)\rangle_\beta = \langle v^*_{\beta+\alpha_\infty}, \mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n).v_\beta\rangle = \prod_{k=1}^n z_k^{2\alpha_k\beta}\prod_{1\le i<j\le n}(z_j-z_i)^{2\alpha_i\alpha_j}. \tag{7.3}$$

7.3. **Solutions to PDEs from vacuum expectations.** Let us then turn our attention to the representations $\mathrm{M}_{c,h}$ of the Virasoro algebra. We consider the case (5.4), with $(c,h) = (c, h_{r,s})$ and $r,s \in \mathbb{Z}_{>0}$, and parametrize the central charge as $c = 1 - 24\alpha_0^2$. It is convenient to denote $\alpha_+ = \sqrt{\kappa}/2$ and $\alpha_- = -2/\sqrt{\kappa}$, and $t = \kappa/4 > 0$, so that $\alpha_+\alpha_- = -1$, and we have $\alpha_0 = \frac{1}{2}(\alpha_+ + \alpha_-)$ and $h(\alpha_\pm) = 1$. Then, one can check that the charges $\alpha_{r,s} := \frac{1}{2}\big((1-r)\alpha_+ + (1-s)\alpha_-\big)$ correspond with $h_{r,s} = h(\alpha_{r,s}) = \alpha_{r,s}^2 - 2\alpha_0\alpha_{r,s}$.

Consider the Virasoro highest weight representation $\mathsf{W}_{c,h_{r,s}} = \mathcal{U}(\mathfrak{Vir}).\mathrm{V}_{\alpha_{r,s}}(z).v_0$ generated by the primary field $\mathrm{V}_{\alpha_{r,s}}(z)$, where $v_0 = v_{\alpha_{1,1}}$. By the universality property, $\mathsf{W}_{c,h_{r,s}}$ is a quotient of the Verma module $\mathrm{M}_{c,h_{r,s}}$ by some submodule $\mathcal{I}$ (recall also Section 6.4). In fact, one can show that the submodule $\mathcal{I}$ contains a singular vector at level rs, using a particular structure on charged Fock spaces known as BRST cohomology [FFK89, Fel89, BMP91]. In particular, if $\alpha_j = \alpha_{r,s}$ for some j, then the vacuum expectation (7.3) with $\beta = 0$ satisfies a PDE associated to this singular vector. For instance, when $\alpha_j = \alpha_{1,s}$, the function (7.3) with $\beta = 0$ is a solution to the Benoit & Saint-Aubin PDE (6.6):

$$\mathcal{D}_s^{(z_j)}\langle \mathrm{V}_{\alpha_1,\ldots,\alpha_n}(z_1,\ldots,z_n)\rangle_0 = 0 \qquad \text{when } \alpha_j = \alpha_{1,s}.$$

To consider applications to the theory of SLE_κ, we remark that, by [BB03a], starting points ξ of SLE_κ curves should correspond with insertions of boundary changing operators $\Psi_{1,2}(\xi)$ of Kac weight $h_{1,2}$. These operators have the degeneracy (5.5) at level two, resulting in PDEs of second order of type (6.6). An example of such a PDE is (4.5), required from the chordal SLE_κ boundary Green's functions. Choose in (7.3) the parameters $\beta = 0$, $\alpha_1 = \alpha_{1,2} = 1/\sqrt{\kappa}$, and $\alpha_{i+1} = \alpha_{1,3} = 2/\sqrt{\kappa}$, for $i = 1, \ldots, n = N$. Then,

$$F(x; y_1, \ldots, y_N) \;=\; \prod_{1 \le l \le N} (y_l - x)^{4/\kappa} \prod_{1 \le i < j \le N} (y_j - y_i)^{8/\kappa}$$

is a solution to the PDE (4.5). However, it does not, in general, satisfy the conformal covariance property (3.9), as one easily checks that the homogeneity degree of the function $F(x; y_1, \ldots, y_N)$ is $\frac{1}{\kappa} 4N(2N-1)$, but under a scaling $z \mapsto \lambda z$, we require $F(\lambda x; \lambda y_1, \ldots, \lambda y_N) = \lambda^{\frac{1}{\kappa}N(\kappa - 8)} F(x; y_1, \ldots, y_N)$, by (3.9). Thus, we need to find another solution. To establish this, we introduce screening charges, which do not affect the intertwining property (7.2) of the vertex operator in (7.3), but change the Heisenberg charge. The idea is that a certain charge neutrality is needed to obtain the correct homogeneity degree in the covariance property (3.9), and this can be established by adding a suitable charge to ∞.

7.4. **Screening.** In order to obtain the correct charge neutrality for the desired conformal covariance, the crucial observation is that the commutators of the primary fields $\Psi_h(z) = \Psi_\pm(z)$ of conformal weight $h = h(\alpha_\pm) = 1$ with the generators L_n of the Virasoro algebra are total derivatives, that is,

$$[\mathrm{L}_n, \Psi_\pm(z)] \;=\; \Big(z^{n+1} \frac{\partial}{\partial z} \;+\; (n+1) z^n \Big) \Psi_\pm(z) \;=\; \frac{\partial}{\partial z} \Big(z^{n+1} \Psi_\pm(z) \Big). \tag{7.4}$$

Therefore, integrating $[\mathrm{L}_n, \Psi_\pm(z)]$ over a closed contour Γ gives zero whenever $z^{n+1}\Psi_\pm(z)$ takes equal values at the endpoints. This implies that we may "screen" the vertex operators $\mathrm{V}_{\alpha_1, \ldots, \alpha_n}$ by the charges $\alpha_\pm$, without changing the intertwining relation (7.2). We thus define the screened vertex operators

$$\begin{aligned} & \mathrm{V}^{\Gamma; \ell_+, \ell_-}_{\alpha_1, \ldots, \alpha_n}(z_1, \ldots, z_n) \\ := & \int_\Gamma \mathrm{V}_{\alpha_1, \ldots, \alpha_n; \alpha_+, \ldots, \alpha_+, \alpha_-, \ldots, \alpha_-}(z_1, \ldots, z_n; u_1, \ldots, u_{\ell_+}, w_1, \ldots, w_{\ell_-}) \, \mathrm{d}u_1 \cdots \mathrm{d}u_{\ell_+} \mathrm{d}w_1 \cdots \mathrm{d}w_{\ell_-}, \end{aligned}$$

for any $\ell_\pm \in \mathbb{Z}_{\ge 0}$. The operators $\mathrm{V}^{\Gamma; \ell_+, \ell_-}_{\alpha_1, \ldots, \alpha_n}$ map the charged Fock space F_{0, α_0} to the space $\hat{\mathrm{F}}_{\alpha_\infty, \alpha_0}$, where $\alpha_\infty = \sum_{i=1}^n \alpha_i + \ell_+ \alpha_+ + \ell_- \alpha_-$. Because the terms with total derivatives vanish, we have the same intertwining relations (7.2) as for the non-screened operators. The vacuum expectations (7.3) become

$$\begin{aligned} & \langle \mathrm{V}^{\Gamma; \ell_+, \ell_-}_{\alpha_1, \ldots, \alpha_n}(z_1, \ldots, z_n) \rangle_0 \;=\; \langle v^*_{\alpha_\infty}, \mathrm{V}^{\Gamma; \ell_+, \ell_-}_{\alpha_1, \ldots, \alpha_n}(z_1, \ldots, z_n).v_0 \rangle \\ = & \int_\Gamma \prod_{1 \le i < j \le n} (z_j - z_i)^{2\alpha_i \alpha_j} \prod_{1 \le r < s \le \ell_+} (u_s - u_r)^{2\alpha_+ \alpha_+} \prod_{1 \le t < u \le \ell_-} (w_u - w_t)^{2\alpha_- \alpha_-} \\ & \times \prod_{\substack{1 \le k \le n \\ 1 \le r \le \ell_+}} (u_r - z_k)^{2\alpha_k \alpha_+} \prod_{\substack{1 \le k \le n \\ 1 \le t \le \ell_-}} (w_t - z_k)^{2\alpha_k \alpha_-} \prod_{\substack{1 \le t \le \ell_- \\ 1 \le r \le \ell_+}} (u_r - w_t)^{2\alpha_- \alpha_+} \; \mathrm{d}u_1 \cdots \mathrm{d}u_{\ell_+} \mathrm{d}w_1 \cdots \mathrm{d}w_{\ell_-}. \end{aligned}$$

7.5. **Application to SLE_κ Green's functions.** Let us return to the question of Green's functions. Taking $\ell_- = N$ and $\ell_+ = 0$, we obtain solutions to both the PDE (4.5) and the covariance (3.9),

$$\begin{aligned} \zeta(x; y_1, y_2, \ldots, y_N) = \int_\Gamma & \prod_{1 \le l \le N} (y_l - x)^{4/\kappa} \prod_{1 \le i < j \le N} (y_j - y_i)^{8/\kappa} \prod_{1 \le t < u \le N} (w_u - w_t)^{8/\kappa} \\ & \times \prod_{1 \le t \le N} (w_t - x)^{-4/\kappa} \prod_{\substack{1 \le i \le N \\ 1 \le u \le N}} (w_u - y_i)^{-8/\kappa} \, \mathrm{d}w_1 \cdots \mathrm{d}w_N, \end{aligned}$$

which, conjecturally, should provide the SLE_κ boundary Green's function with a suitable choice of the integration surface Γ (see also [B] and Section 8.2). Note that the charge at the point ∞ is non-zero: $\alpha_\infty = N \alpha_{1,3} + \alpha_{1,2} + \ell_- \alpha_- = \alpha_{1,2}$. Indeed, the function ζ needs not transform covariantly under special conformal transformations (e.g., inversions $z \mapsto 1/z$), as the covariance property (3.9) for $\Lambda = \mathbb{H}$ only

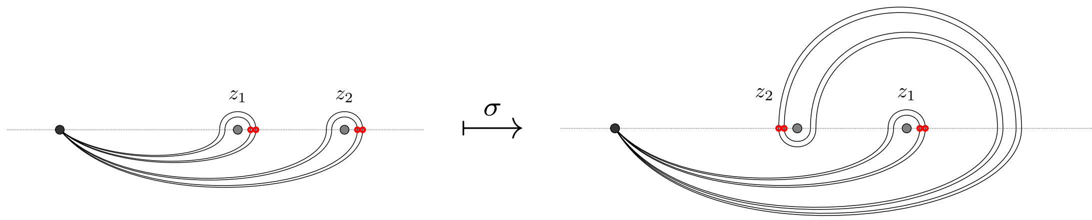


FIGURE 7.1. This figure illustrates the contour deformation method which can be used to calculate the monodromy of functions of Coulomb gas type, with integration contours given in terms of the contours depicted in Figure 8.1 in the next section.

concerns Möbius maps that fix the point ∞. This is natural, as ∞ is a distinguished point — it is the end point of the SLE_κ curve, carrying the boundary changing operator $\Psi_{1,2}(\infty)$.

7.6. **Application to multiple SLE_κ.** The required PDE system (8.5) for multiple SLE_κ is also an example of the degeneracy of $\Psi_{1,2}$ at level two. Choosing $\alpha_i = \alpha_{1,2} = 1/\sqrt{\kappa}$ for all i, and the screenings $\ell_- = N$ and $\ell_+ = 0$, from the vacuum expectation (7.3) with $\beta = 0$ one obtains a solution to (8.5),

$$\mathcal{Z}(z_1, \ldots, z_{2N}) = \int_\Gamma \prod_{1\leq i<j\leq 2N} (z_j - z_i)^{2/\kappa} \prod_{1\leq r<s\leq N} (w_s - w_r)^{8/\kappa} \prod_{\substack{1\leq i\leq 2N \\ 1\leq r\leq N}} (w_r - z_i)^{-4/\kappa} \; \mathrm{d}w_1 \cdots \mathrm{d}w_N,$$

which, conjecturally, should provide multiple SLE_κ partition functions with suitable choices of the integration surfaces Γ (see also [B] and Section 8.3). Note also that we have $\alpha_\infty = 2N\alpha_i + N\alpha_- = 0$, so there is no charge at infinity. In fact, the screened vertex operator $\mathrm{V}^{\Gamma;N,0}_{\alpha_1,\ldots,\alpha_n}$ has a fully Möbius covariant vacuum expectation $\mathcal{Z}(z_1, \ldots, z_{2N})$, see Equation (8.6), and e.g. [A, Theorem 4.17] for the proof.

7.7. **Integration surfaces.** For a small number of variables, explicit calculations of the monodromy of functions of the above type were performed in [DF84, DF85] using a contour deformation method, illustrated in Figure 7.1. However, the method becomes cumbersome when the number or variables increases. There is another, more systematic method for considering the monodromy properties — a hidden quantum group symmetry [FW91, GRAS96, A]. Originally, such a hidden structure was discovered by Vladimir Drinfeld and Toshitake Kohno [Koh87, Dri90] in the context of the Knizhnik-Zamolodchikov equations. It was then noticed [FFK89, FW91] that the same method yields a topological action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ on the integration surfaces Γ that are used to construct solutions to PDEs such as (6.6) in terms of screened vertex operators. We exploit this strategy in the article [G].

The operator $\mathscr{R}$ defined in (5.8) gives a braided structure on tensor product representations of the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$. On the other hand, the action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the integration surfaces Γ is in fact isomorphic to a tensor product representation, see [FW91, Pel12, A]. In particular, the two representations of the braid group defined, on one hand, by the monodromy of the Coulomb gas type solutions of the Benoit & Saint Aubin PDEs (6.6), and on the other hand, by the operator $\mathscr{R}$, are isomorphic. After the discovery of this hidden symmetry, it was possible to identify the so called conformal blocks of CFT, correlation functions that diagonalize the monodromy action [FFK89, Fel89]. In [D, E, F, G], we explicitly construct the unique monodromy invariant solution to the Benoit & Saint Aubin PDEs, in terms of the conformal blocks, using the quantum group method (see also Section 8.6 for the uniqueness).

To find a solution to the PDEs (6.6) in integral form, one has to construct a closed integration surface Γ in order to use the observation (7.4). It is not difficult to find examples of closed surfaces, and one can study a homology theory for them [FW91, TK86], but it is hard to analyze the asymptotic properties of integrals over complicated surfaces. This can, nevertheless, be established via the hidden quantum group symmetry, with a clever choice of basis for the surfaces Γ, depicted in Figure 8.1 in Section 8.1. Importantly, the method allows one to formulate PDE boundary value problems as linear systems of equations in a tensor product representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ and, in particular, to obtain explicit solutions.

Contribution of this thesis

In this last part, we discuss the contributions of this thesis to random geometry: the theory of SLEs, CFT, and lattice models. We begin with a summary of the quantum group method developed in the article [A]. We then discuss applications to SLE_κ boundary Green's functions, multiple SLEs, and the monodromy properties of CFT correlation functions, studied in [B, D, E, F, G]. We also outline scaling limit results obtained in [C], concerning connectivity and boundary visit probabilities of the planar UST and LERW. Interestingly, these scaling limits are solutions to PDE systems arising from CFT.

8. Description of the main results

8.1. **The "Spin chain – Coulomb gas correspondence" method.** Let $\mathcal{D}^{(z_j)}_{d_j}$ be the Benoit & Saint-Aubin differential operator (6.7) with $t = \kappa/4 > 0$, discussed in Section 6.4, and let $\mathfrak{W}_n$ denote the configuration space (6.1), as in Section 6.2. We consider solutions $F \colon \mathfrak{W}_n \to \mathbb{C}$ to the PDE system

$$\mathcal{D}^{(z_j)}_{d_j} F(z_1, \ldots, z_n) \; = \; 0 \qquad \text{for all} \;\; j = 1, \ldots, n. \tag{8.1}$$

In the article [A], we exploit the hidden quantum group symmetry of CFT to develop a systematic method for solving the PDEs (8.1), with specific boundary conditions given in terms of asymptotic behavior of the functions. The method is based on the representation theory of the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, in the generic case when the parameter $q = e^{\mathrm{i}\pi 4/\kappa}$ is not a root of unity (i.e., κ is irrational).

The key ingredient of the quantum group method is the "Spin chain – Coulomb gas correspondence", a linear mapping $v \mapsto \mathcal{F}[v]$ which associates vectors $v \in \mathsf{V} = \mathsf{M}_{d_n} \otimes \cdots \otimes \mathsf{M}_{d_1}$ in a tensor product of d_i-dimensional irreducible representations M_{d_i} of $\mathcal{U}_q(\mathfrak{sl}_2)$ to integral functions $\mathcal{F}[v](x_1, \ldots, x_n)$ of Coulomb gas type. Importantly, properties of the functions are encoded in natural, representation theoretical properties of the vectors. Solutions to the PDEs (8.1) correspond with highest weight vectors, that is, vectors annihilated by the generator $E \in \mathcal{U}_q(\mathfrak{sl}_2)$. The conformal covariance of the function $\mathcal{F}[v]$ is related to the eigenvalue of the vector v under the Cartan subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$, generated by the element $K \in \mathcal{U}_q(\mathfrak{sl}_2)$. Finally, the asymptotics of $\mathcal{F}[v]$ correspond with projections of v onto subrepresentations.

Theorem. [A, Theorem 4.17] *Suppose q is not a root of unity or zero. Let $v \in \mathsf{V}$ satisfy $E.v = 0$ and $K.v = q^{\lambda - 1}\, v$. The function $\mathcal{F}[v] : \big\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\big|\; x_1 < \cdots < x_n \big\} \to \mathbb{C}$ has the following properties.*

- **(PDE):** *It satisfies the PDE system* (8.1).
- **(COV):** *It is translation invariant and homogeneous of degree that depends on λ. When $\lambda = 1$, then $\mathcal{F}[v]$ transforms covariantly under all Möbius transformations, i.e., property* (6.2) *holds.*
- **(ASY):** *Suppose that v belongs to the subrepresentation of $\mathsf{V} = \mathsf{M}_{d_n} \otimes \cdots \otimes \mathsf{M}_{d_1}$ obtained by picking the d-dimensional irreducible direct summand in the tensor product of the j:th and $j+1$:st factors,*

$$\mathsf{M}_{d_j} \otimes \mathsf{M}_{d_{j+1}} \;\; \cong \;\; \mathsf{M}_{d_j + d_{j+1} - 1} \oplus \mathsf{M}_{d_j + d_{j+1} - 3} \oplus \cdots \oplus \mathsf{M}_{|d_j - d_{j+1}| + 3} \oplus \mathsf{M}_{|d_j - d_{j+1}| + 1}.$$

Let $\hat{v}$ denote the vector obtained by identifying the vector v as an element in the shorter tensor product $\Big(\bigotimes_{i=j+2}^{n} \mathsf{M}_{d_i} \Big) \otimes \mathsf{M}_d \otimes \Big(\bigotimes_{i=1}^{j-1} \mathsf{M}_{d_i} \Big)$. Then, the function $\mathcal{F}[v]$ has the power law behavior

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{F}[v](x_1, \ldots, x_n)}{|x_{j+1} - x_j|^{\Delta_d}} \; = \; B \times \mathcal{F}[\hat{v}](x_1, \ldots, x_{j-1}, \xi, x_{j+2} \ldots, x_n),$$

where the constant $B = B^{d_j, d_{j+1}}_d$ and exponent $\Delta_d = \Delta^{d_j, d_{j+1}}_d$ are explicitly given in [A].

Explicitly, for a vector $v \in \mathsf{V}$, the function $\mathcal{F}[v]$ is of the Coulomb gas form (recall also Section 7)

$$\mathcal{F}[v](x_1, \ldots, x_n) \; = \; \int_{\Gamma(v)} f^{(\ell)}(\boldsymbol{x}; \boldsymbol{w}) \, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell, \qquad \text{where} \tag{8.2}$$

$$f^{(\ell)}(\boldsymbol{x}; \boldsymbol{w}) \; = \; \prod_{1 \le i < j \le n} (x_j - x_i)^{2\alpha_i \alpha_j} \prod_{\substack{1 \le i \le n \\ 1 \le r \le \ell}} (w_r - x_i)^{2\alpha_i \alpha_-} \prod_{1 \le r < s \le \ell} (w_s - w_r)^{2\alpha_- \alpha_-}, \tag{8.3}$$

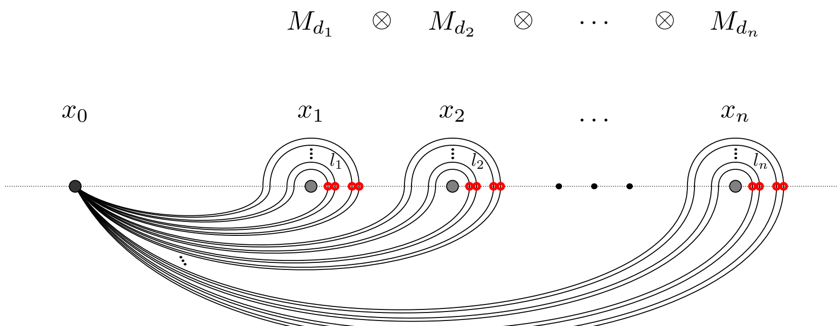


FIGURE 8.1. For certain basis vectors $e_{l_1,\ldots,l_n} = e_{l_n} \otimes \cdots \otimes e_{l_1} \in \mathsf{M}_{d_n} \otimes \cdots \otimes \mathsf{M}_{d_1}$, the integration ℓ-surface $\Gamma(e_{l_1,\ldots,l_n}) = \mathfrak{L}^{\ni}_{l_1,\ldots,l_n}$ with $\ell = \sum_{i=1}^n l_n$ is depicted in this figure. It depends on an auxiliary point x_0 such that $x_0 < x_1 < x_2 < \ldots < x_n$. The integration contours associated to linear combinations of the basis vectors $e_{l_1,\ldots,l_n}$ are linear combinations of $\mathfrak{L}^{\ni}_{l_1,\ldots,l_n}$. By [A, Proposition 4.5], for highest weight vectors v, the contour $\Gamma(v)$ is homologically closed and independent of x_0. The branch of the multivalued integrand (8.3) is chosen such that $f^{(\ell)}(\cdot;\boldsymbol{w})$ is positive at the point $\boldsymbol{w} = (w_1, \ldots, w_\ell)$, marked by red circles. (Unfortunately, in our convention, the order of the tensor components is chosen opposite to the order illustrated in this figure).

the powers are chosen as $\alpha_i = \frac{d_i - 1}{\sqrt{\kappa}}$ and $\alpha_- = -\frac{2}{\sqrt{\kappa}}$, and the integration contours $\Gamma(v)$ are ℓ-surfaces which can be written as linear combinations of basis elements corresponding to the natural basis of the tensor product representation $\mathsf{V} = \mathsf{M}_{d_n} \otimes \cdots \otimes \mathsf{M}_{d_1}$, as explained in Figure 8.1, and in [A] in detail.

The proof of the above theorem constitutes the whole article [A]. We summarize the main ideas below.

Idea of the proof. For any closed surface $\Gamma(v)$, one can directly show that the integral function (8.2) satisfies the PDE system (8.1), by using an integration by parts formula and the property that the integrand (8.3) is chosen in such a way that $\mathcal{D}^{(x_j)}_{d_j} f^{(\ell)}$ is a "total derivative" [A, Corollaries 4.8 and 4.11], in the spirit of the observation (7.4) in Section 7. "Closedness" of $\Gamma(v)$ is characterized by the property that it corresponds to a highest weight vector[30], that is, $E.v = 0$. For property (COV), we prove in [A, Proposition 4.15], that if $\ell = \frac{1}{2}\sum_{i=1}^n (d_i - 1)$, then the function (8.2) is Möbius covariant. This corresponds with the eigenvalue equation $K.v = v$. When the K-eigenvalue of v is not equal to one, the homogeneity property is easy to see. Finally, property (ASY) is illustrated in Figure 8.2. It is proved by isolating the contours surrounding the variables x_j and x_{j+1}, see [A, Proposition 4.4] for details.

8.2. **SLE boundary visits.** Chordal SLE$_\kappa$ boundary Green's functions are defined as renormalized probability amplitudes for the SLE$_\kappa$ curve γ to pass through small neighborhoods of given boundary points (recall Section 3.9). By conformal invariance of SLE$_\kappa$, it suffices to consider the case of the upper half-plane $\mathbb{H}$. We fix the order of visits to the given boundary points $y_1, \ldots, y_N \in \mathbb{R}$, denoted by ω as in [B], and denote the limit of the probability of the ordered visiting event by

$$\lim_{\varepsilon_1,\ldots,\varepsilon_N \searrow 0} (\varepsilon_1 \cdots \varepsilon_N)^{1-8/\kappa}\, \mathsf{P}^{(\mathbb{H};x,\infty)}\big[\, \gamma \text{ comes } \varepsilon_i\text{-close to } y_i \text{ for all } i \text{ in the given order } \omega \,\big] \\ =: \text{const.} \times \zeta_\omega(x; y_1, \ldots, y_N). \tag{8.4}$$

The general probability amplitude $\zeta(x; y_1, \ldots, y_N)$ is then given by summing over all possible orders.

The function ζ_ω satisfies the conformal covariance (3.9) for Möbius maps fixing the point ∞. From the vanishing Itô drift of the SLE$_\kappa$ local martingale (4.4) associated to ζ_ω, we find that ζ_ω has to satisfy the PDE (4.5). This PDE is of Benoit & Saint-Aubin type, associated to the degeneracy of $\Psi_{1,2}(x)$ at level two (which gives a heuristic argument why ζ_ω should solve the PDE). Similarly, we also expect that

[30] The generator $E \in \mathcal{U}_q(\mathfrak{sl}_2)$ can be thought of as a boundary operator in a suitable homology, see [FW91].

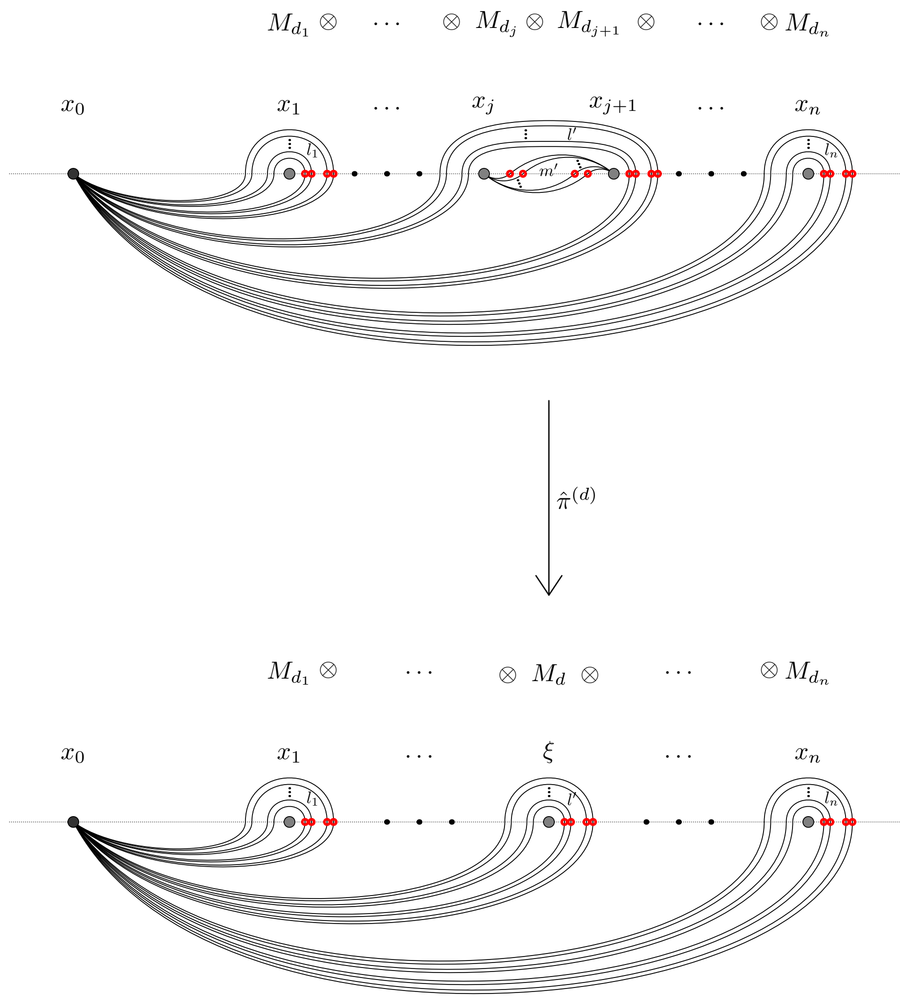


FIGURE 8.2. This figure illustrates how to isolate the behavior of the functions as the consecutive variables x_j, x_{j+1} tend together. The symbol $\hat{\pi}^{(d)}$ denotes the projection picking the d-dimensional irreducible direct summand M_d in the Clebsch-Gordan type decomposition of the tensor product of the j:th and $(j+1)$:st factors $\mathsf{M}_{d_{j+1}} \otimes \mathsf{M}_{d_j}$. With this notation, we have $\hat{\pi}^{(d)}(v) = \hat{v} \in \left(\bigotimes_{i=j+2}^{n} \mathsf{M}_{d_i} \right) \otimes \mathsf{M}_d \otimes \left(\bigotimes_{i=1}^{j-1} \mathsf{M}_{d_i} \right)$ as in (ASY). (The order of the tensor components is again opposite to the order in the figure).

ζ_ω satisfies N third order PDEs[31] $\mathcal{D}_3^{(y_i)} \zeta_\omega(x; y_1, \ldots, y_N) = 0$, for all i, associated to the degeneracies of $\Psi_{1,3}(y_i)$ at level three, where the partial differential operators $\mathcal{D}_3^{(y_i)}$ are given in (6.7), with the conformal weight $h_{1,2} = \frac{6-\kappa}{2\kappa}$ for the starting point x of the curve, and $h_{1,3} = \frac{8-\kappa}{\kappa}$ for the visited points y_i.

Now, to single out the solutions corresponding to the event of boundary visits in the given order ω, we need to impose boundary conditions, similarly as in Section 4.4. These are given by specific asymptotic behavior of ζ_ω, similarly as (4.6) — see also Figure 4.1. The boundary conditions can be deduced from the qualitative properties of the visiting probability, when either two nearby points y_i and y_{i+1} approach each other, or a point y_k and the starting point x of the curve approach each other. For example, if the points y_i, y_{i+1} are not to be successively visited in the given visiting order ω, the boundary visit probability should (morally) vanish when $|y_{i+1} - y_i| \to 0$, whereas if they are to be successively visited,

[31]The probabilistic interpretation of these third order PDEs, however, is not clear.

then as $|y_{i+1} - y_i| \to 0$, the probability amplitude ζ_ω should have a similar form but with one variable less, as in Equation (4.6). The detailed boundary conditions are given in the article [B].

The following result is proved in [B] using the correspondence, with M_2 at x, and M_3 at y_i for $i = 1, \ldots, N$.

Theorem. [B, Theorem 5.2] *For $\kappa \in (0,8) \setminus \mathbb{Q}$, there exists a unique collection $(\widetilde{\zeta}_\omega)$ of functions which satisfy the second order PDE* (4.5)*, third order PDEs $\mathcal{D}_3^{(y_i)} \widetilde{\zeta}_\omega = 0$ for all $i = 1, \ldots, N$, covariance* (3.9)*, and recursive boundary conditions specified by the visiting orders ω, given in* [B] *in detail.*

In the articles [D, E], we prove that the solutions specified by different orders of visits are linearly independent (when $\kappa \notin \mathbb{Q}$), and also that the dimension of the image of the mapping $\mathcal{F}$ is equal to the cardinality of the set of link patterns of certain type. Uniqueness of solutions is established in this space. We conjecture[32] that the solutions $\widetilde{\zeta}_\omega$ are the SLE_κ boundary visit probability amplitudes ζ_ω of (8.4).

8.3. **Pure partition functions of multiple SLE_κ.** The main objective of the article [B] is the construction of multiple SLE_κ processes. Recall from Section 3.7 how partition functions arise in the growth process construction of a local multiple SLE_κ, see also [BBK05, Dub07a, Gra07]. The partition functions $\mathcal{Z}$ are defined for $x_1 < \ldots < x_{2N}$, and they are subject to a number of requirements:

- **Partial differential equations of second order:**

$$\left[\frac{\kappa}{2} \frac{\partial^2}{\partial x_i^2} + \sum_{j \neq i} \left(\frac{2}{x_j - x_i} \frac{\partial}{\partial x_j} - \frac{2h_{1,2}}{(x_j - x_i)^2} \right) \right] \mathcal{Z}(x_1, \ldots, x_{2N}) = 0 \qquad \text{for all } i = 1, \ldots, 2N. \tag{8.5}$$

- **Conformal covariance:** for all Möbius maps $\mu \colon \mathbb{H} \to \mathbb{H}$ such that $\mu(x_1) < \ldots < \mu(x_{2N})$,

$$\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \mu'(x_i)^{h_{1,2}} \times \mathcal{Z}(\mu(x_1), \ldots, \mu(x_{2N})). \tag{8.6}$$

- **Positivity:** $\mathcal{Z}(x_1, \ldots, x_{2N}) > 0$ for all $(x_1, \ldots, x_{2N}) \in \mathfrak{X}_{2N}$.

A local N-SLE_κ can be thought of as a process of initial segments of $2N$ SLE_κ curves, started from disjoint boundary points and stopped before the curves meet, see [B, Appendix A] for details. We prove in [B] that the local N-SLE_κ measures are classified by the partition functions $\mathcal{Z}$ satisfying the above three requirements — see also [Dub07a]. Furthermore, we prove that the convex set of local multiple SLE_κ probability measures is in one-to-one correspondence with the set of (normalized) partition functions, and the convex structures of the two sets agree. We also construct a particular basis for solutions of (8.5) – (8.6), the pure partition functions $\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})$, labeled by planar pair partitions α of $2N$ points. These functions should correspond to the extremal measures of the multiple SLE_κ processes having deterministic pairwise connectivities of the random curves (recall Figure 2.5).

Theorem. [B, Theorems A.4 and 4.1]

- **(i) :** *Any partition function $\mathcal{Z}$ can be used to construct a local multiple* SLE_κ*, and two functions $\mathcal{Z}, \tilde{\mathcal{Z}}$ give rise to the same local multiple* SLE_κ *if and only if they are constant multiples of each other.*
- **(ii) :** *Any local multiple* SLE_κ *can be constructed from a unique (normalized) partition function.*
- **(iii):** *If $\mathcal{Z} = r\mathcal{Z}_1 + (1-r)\mathcal{Z}_2$ is a convex combination of two partition functions, $0 \le r \le 1$, then the local multiple* SLE_κ *probability measures associated to $\mathcal{Z}$ are convex combinations of the probability measures associated to $\mathcal{Z}_1$ and $\mathcal{Z}_2$, with coefficients proportional to r and $1 - r$, and proportionality constants depending on the conformal moduli of the domain with the marked points.*
- **(iv):** *For any $\kappa \in (0,8) \setminus \mathbb{Q}$, there exists a collection $(\mathcal{Z}_\alpha)$ of functions, indexed by planar pair partitions α of any even number $2N$ of points, satisfying* (8.5) – (8.6)*, and the requirements* (8.7) *below.*
- **(v) :** *For any fixed $N \in \mathbb{Z}_{\geq 0}$, the set $\{\mathcal{Z}_{\alpha_i} \mid i = 1, \ldots, \mathrm{C}_N\}$ is linearly independent and it spans a space of solutions to* (8.5) – (8.6) *of dimension*[33] $\mathrm{C}_N = \frac{1}{N+1}\binom{N}{2N}$.

[32] To prove this, a strategy could be to use optional stopping similarly as in the example in Section 4.4. A sufficient control of the functions $\widetilde{\zeta}_\omega$ is needed to obtain uniform integrability of the local martingales.

[33] Recall that the number of planar pair partitions α of $2N$ points is the Catalan number C_N.

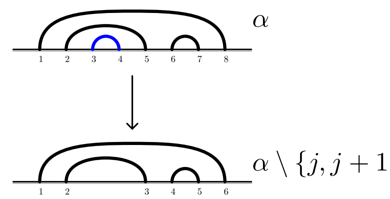
FIGURE 8.3. Illustration of the boundary conditions (8.7) for the multiple SLE_κ pure partition functions $\mathcal{Z}_\alpha$. A non-zero solution $\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})$ to (8.5) – (8.6) has exactly one of the two possible asymptotics [FK15d, Theorem 2]:

$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \sim (x_{j+1} - x_j)^{\frac{\kappa-6}{\kappa}} \qquad \text{or} \qquad \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \sim (x_{j+1} - x_j)^{\frac{2}{\kappa}}.$$

The asymptotics on the left and right, respectively, should correspond with the behavior of the curves depicted on the left and right in the figure. If the two curves meet, it is natural to require a cascade property for the behavior of the other curves, as in Figure 3.3 and Equation (8.7). The conditions (8.7) state that after removing the interface formed by the j:th and $(j+1)$:st curves, the partition function of the process consisting of the remaining curves is given by $\mathcal{Z}_{\alpha \setminus \{j,j+1\}}$, see also [B, Proposition A.6].

The proof of (i), (ii) & (iii) relies on the results of [Dub07a]. In the proof of (iv) & (v), we use the "Spin chain – Coulomb gas correspondence" with $\mathsf{M}_2^{\otimes 2N}$. The pure partition functions are singled out by imposing the following recursive boundary conditions: for any $j = 1, \ldots, 2N-1$, we have

$$\text{(8.7)} \qquad \lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{\frac{\kappa-6}{\kappa}}} = \begin{cases} 0 & \text{if } \{j, j+1\} \notin \alpha \\ \mathcal{Z}_{\alpha \setminus \{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha. \end{cases}$$



In Figure 8.3, we intuitively describe the meaning of the boundary conditions (8.7). We expect the basis functions $\mathcal{Z}_\alpha$ to correspond with the deterministic connectivities of the curves, given by the planar pair partitions α. The pair partition α should therefore determine whether or not the two curves meet.

We conjecture that the normalization can be chosen so that all the functions $\mathcal{Z}_\alpha$ are simultaneously positive. This would imply that $\mathcal{Z}_\alpha$ determine the convex set of the local SLE_κ probability measures.

8.4. **General solution space of the Benoit & Saint-Aubin PDEs.** Solutions of the equations (8.5) – (8.6) have also been studied by Steven Flores and Peter Kleban in [FK15a, FK15b, FK15c, FK15d]. They proved that the dimension of the solution space of (8.5) – (8.6), subject to a power law bound on the growth of the functions, equals the Catalan number C_N. Our functions $\mathcal{Z}_\alpha$ thus form a basis of this space, and are unique within the space of solutions growing at most as a power law.

With Flores, we consider solutions to the general Benoit & Saint-Aubin PDE systems (8.1) in the articles [D, E]. We prove in [E] that the correspondence map $\mathcal{F}$ is a linear isomorphism onto the space of solutions of (8.1) obtained as suitable limits of the solutions of the second order PDE (8.5) — in the spirit of fusion of CFT. Furthermore, we prove that the solution space thus obtained has the conjectured full dimension, the number of certain link patterns (generalizing C_N for the planar pair partitions).

In [D, Theorem 5.3], we construct a particular basis for solutions to the Benoit & Saint-Aubin PDEs (8.1), with specific asymptotic boundary conditions, motivated by fusion of CFT. The boundary conditions generalize the recursive properties (8.7) for general link patterns exemplified in Figure 8.4. The basis functions are labeled by such link patterns, and the asymptotics are given in terms of removing links. These functions should be the pure partition functions of multiple SLE_κ with packets of curves with common starting points. This basis is crucial in analyzing the solution space.

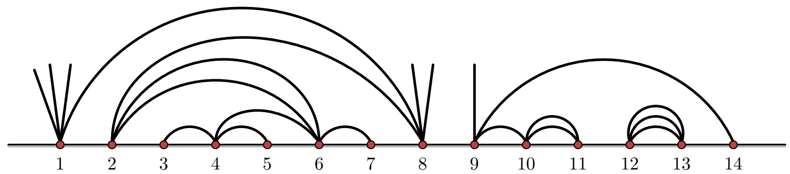


FIGURE 8.4. Illustration of a connectivity where several curves start from the same point.

8.5. **Connectivity and boundary visit probabilities of branches of UST.** In the article [C], we find explicit expressions for probabilities of connectivity and boundary visit events for the planar LERW and UST — see Figures 8.5 and 8.6 for illustrations of such events. The connectivity probabilities are formulated as generalizations of Fomin's formula (2.2), in terms of sums of determinants of Fomin type with explicit combinatorial coefficients. The boundary visit probabilities are obtained from the connectivity probabilities via a simple measure preserving bijection. We refer to [C] for details.

Furthermore, we prove that these probabilities, when suitably renormalized, converge in the scaling limit to conformally covariant functions which satisfy the same Benoit & Saint-Aubin type PDEs as the SLE_2 pure partition functions (for connectivity probabilities), or the chordal SLE_2 boundary Green's functions (for boundary visit probabilities). Our results also give a construction of a local multiple SLE_2.

Theorem. [C, Theorems 3.14, 3.15, 4.1, and 4.2] *Fix a planar pair partition α. For the UST with wired boundary conditions, the probability* (2.1) *of the connectivity event described by α, as illustrated in Figure 8.6, has an explicit determinantal formula. In the square grid approximation of a continuum domain Λ, as detailed in* [C], *the connectivity probability has the conformally covariant scaling limit*

$$\frac{1}{\delta^{2N}}\,\mathbb{P}[\,\xi_i \text{ connects to } \overline{\eta}_i \text{ for all } i=1,\ldots,2N\,] \quad \underset{\delta\to 0}{\longrightarrow} \quad \frac{1}{\pi^N}\times\prod_{j=1}^{2N}|\phi'(p_j)|\times\mathcal{Z}_\alpha\big(\phi(p_1),\ldots,\phi(p_{2N})\big),$$

where $\phi\colon\Lambda\to\mathbb{H}$ is a conformal map, and the function $\mathcal{Z}_\alpha$ has an explicit determinantal formula. The function $\mathcal{Z}_\alpha$ is positive and it satisfies the PDE system (8.5), *Möbius covariance* (8.6), *and asymptotics* (8.7) *with $\kappa=2$. In particular, there exists a local multiple* SLE_2 *with the partition function $\mathcal{Z}_\alpha$.*

The proof of the scaling limit result relies on the explicit combinatorial formulas for the probabilities in the discrete models. This enables us to control the functions sufficiently well in the scaling limit, using known convergence results for the discrete harmonic measures to the continuum ones, together with suitable cancellations and modifications in the combinatorial expressions — see [C] for details.

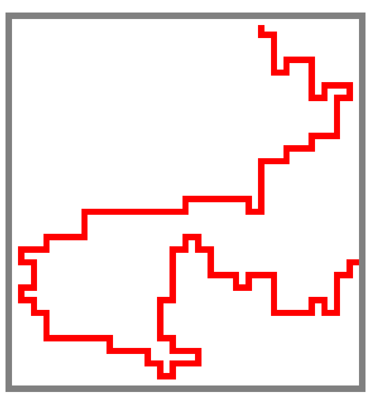


FIGURE 8.5. Illustration of a branch of the uniform spanning tree, or equivalently, a loop-erased random walk, passing through edges at unit distance from the boundary. We call this type of events boundary visit events.

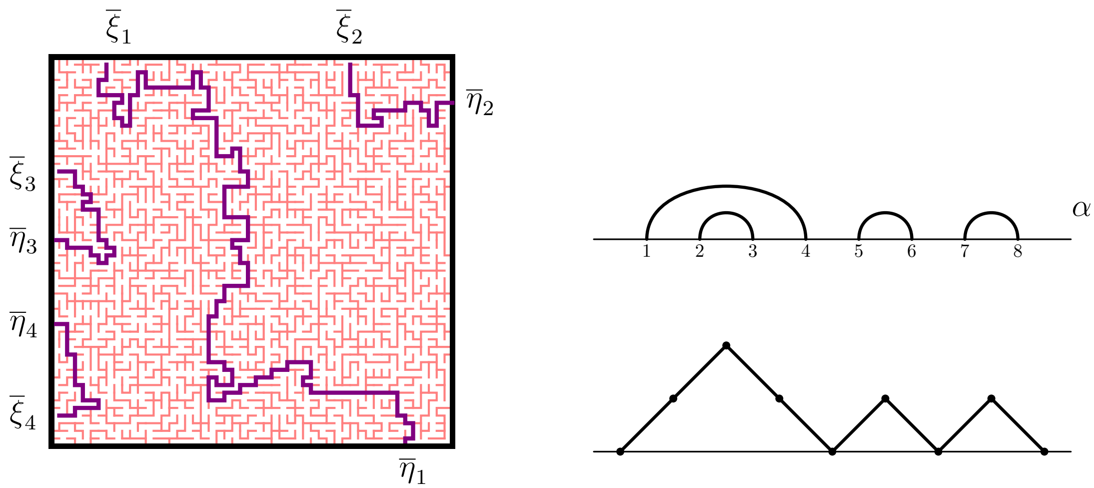


FIGURE 8.6. Example of a connectivity event of boundary branches of the uniform spanning tree and an illustration of how it can be described by a planar pair partition α. In the scaling limit, as the lattice spacing of the square grid approximation of a continuum domain Λ tends to zero (see [C] for the detailed formulation), the boundary edges $\overline{\eta}_1, \overline{\eta}_2, \overline{\xi}_2, \overline{\xi}_1, \overline{\xi}_3, \overline{\eta}_3, \overline{\eta}_4, \overline{\xi}_4$ converge to boundary points $p_1, \ldots, p_8 \in \partial\Lambda$.

In [C, Theorem 3.17], a similar result is proved for the probability of any boundary visit event of the LERW, as illustrated in Figure 8.5 (we omit the precise formulation here, and refer to [C] for details). Interestingly, the scaling limit is a conformally covariant solution to the PDE (4.5) and third order PDEs of type $\mathcal{D}_3^{(y_i)} F = 0$. This is the same PDE system that the chordal SLE_2 boundary Green's function of Section 3.9 is expected to satisfy, and we conjecture that this scaling limit is the Green's function ζ with $\kappa = 2$ (recall also from Section 3.5 that the LERW converges in the scaling limit to SLE_2). Moreover, specifying the order ω of visits as in Section 8.2 and [B], the scaling limits should give the chordal SLE_2 boundary visit probability amplitudes ζ_ω of (8.4) with $\kappa = 2$. Indeed, the asymptotic boundary conditions for ζ_ω, described in Section 8.2, can be explicitly verified for the scaling limit functions.

8.6. **Monodromy invariance of CFT correlation functions.** The functions obtained via the correspondence map $\mathcal{F}$ are, a priori, defined for real variables $x_1 < \ldots < x_n$, but they can be analytically continued to multi-valued functions $\mathcal{F}[v](z_1, \ldots, z_n)$ on the configuration space $\mathfrak{W}_n$, with singularities only at coinciding variables, $z_i = z_j$ for $i \neq j$. Our objective in the series of articles [D, E, F, G] is to show that there exists a unique single-valued solution to the PDE system (8.1) involving both the holomorphic and anti-holomorphic sectors: we take $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ instead of $(z_1, \ldots, z_n)$ in (8.1).

By the properties (PDE) & (COV), given in Section 8.1, the correspondence $\mathcal{F}$ associates highest weight vectors $v \in \mathsf{V} = \mathsf{M}_{d_n} \otimes \cdots \otimes \mathsf{M}_{d_1}$ of Cartan eigenvalue one to Möbius covariant solutions $\mathcal{F}[v](z_1, \ldots, z_n)$ of the Benoit & Saint-Aubin type PDEs (8.1). We denote the set of such vectors by

$$\mathsf{H}_1 \; = \; \mathsf{H}_1(\mathsf{V}) \; = \; \{v \in \mathsf{V} \mid E.v = 0, \; K.v = v\}.$$

Recall from Section 5.8 that tensor product representations of braided Hopf algebras also carry a representation of the pure braid group $\mathfrak{PBr}_n$. Consider the braiding defined by the operator $\mathscr{R}$ of (5.8) on the tensor product $\mathsf{V} = \mathsf{M}_{d_n} \otimes \cdots \otimes \mathsf{M}_{d_1}$ of irreducible representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. By [FW91, Pel12], the correspondence map $v \mapsto \mathcal{F}[v]$ intertwines with the braiding in the sense that

$$\sigma.\mathcal{F}[v](z_1, \ldots, z_n) \; = \; \mathcal{F}[\sigma.v](z_1, \ldots, z_n) \qquad \text{for all } \sigma \in \mathfrak{Br}_n \text{ and } v \in \mathsf{V}, \tag{8.8}$$

where the braiding on the left hand side is given by the monodromy of the function $\mathcal{F}[v](z_1, \ldots, z_n)$ in analytic continuation around the singularities $z_i = z_j$ for $i \neq j$, and the braiding on the right hand side is given by the operator $\mathscr{R}$, that is, $\sigma_i.v = \mathscr{R}_i(v)$ for the braid group generators $\sigma_1, \ldots, \sigma_{n-1} \in \mathfrak{Br}_n$.

Denote by $\overline{\mathsf{V}}$ the conjugate representation of the pure braid group $\mathfrak{PBr}_n$, in which the braiding is performed by $\sigma_i \mapsto \overline{\mathscr{R}}_i$, where $\overline{\mathscr{R}}$ denotes the operator (5.8) with q replaced by q^{-1}, and the quantum group acts as usual. The action of $\mathfrak{PBr}_n$ on $\mathsf{H}_1 \otimes \overline{\mathsf{H}}_1$ thus defined is given by $\sigma_i.(v \otimes w) \;=\; \mathscr{R}_i(v) \otimes \overline{\mathscr{R}}_i(w)$.

Taking into account also the anti-holomorphic sector, we consider functions of the form

$$F(z_1,\ldots,z_n) \;=\; \sum_{k,l} \mathcal{F}[v_k](z_1,\ldots,z_n)\, \mathcal{F}[v_l](\bar{z}_1,\ldots,\bar{z}_n) \;=\; \sum_{k,l} (\mathcal{F}[v_k] \otimes \mathcal{F}[v_l])(z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)$$

for some $v_k \in \mathsf{H}_1$, $v_l \in \overline{\mathsf{H}_1}$. Their monodromy is defined counterclockwise for the holomorphic parts $\mathcal{F}[v_k](z_1,\ldots,z_n)$ and clockwise for the anti-holomorphic parts $\mathcal{F}[v_l](\bar{z}_1,\ldots,\bar{z}_n)$. In general, we seek solutions $(\mathcal{F} \otimes \mathcal{F})[v](z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)$ to the PDEs (8.1) which are invariant under the pure braids $\sigma \in \mathfrak{PBr}_n$, to preserve the order of the variables $z_1,\ldots,z_n$ (i.e., the analytic continuation is performed around a full twist). By the intertwining property (8.8), the monodromy of $(\mathcal{F} \otimes \mathcal{F})[v]$ translates into the braiding of the vector $v = \sum_{k,l} v_k \otimes v_l \in \mathsf{H}_1 \otimes \overline{\mathsf{H}}_1$.

Theorem. [G] *Up to normalization, there exists a unique solution*

$$F(z_1,\ldots,z_n) = (\mathcal{F} \otimes \mathcal{F})[v](z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n), \qquad \textit{with} \quad v \in \mathsf{H}_1 \otimes \overline{\mathsf{H}}_1, \tag{8.9}$$

to the PDE system (8.1) *with the variables* $(z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)$*, such that* F *is Möbius covariant as in* (6.2)*, and monodromy invariant, that is, single-valued for* $(z_1,\ldots,z_n) \in \mathfrak{W}_n$.

Outline of the proof. The proof of this result will be given in [G], using the results obtained in the series of articles [D, E, F]. Basically, it has two ingredients — an invariant bilinear form and Schur's lemma. There is a non-degenerate invariant bilinear form $(\cdot,\cdot)$ on H_1, satisfying

$$(\sigma_i.v, \sigma_i^{-1}.w) \;=\; \big(\mathscr{R}_i(v), \mathscr{R}_i^{-1}(w)\big) \;=\; (v,w) \qquad \text{for all } v,w \in \mathsf{H}_1,\; i = 1,\ldots,n-1.$$

It is easy to construct such an invariant bilinear form, but the nondegeneracy is more difficult to show [F]. The application of Schur's lemma becomes clear below in the last step of the proof (Step 4).

Step 1. Observe that, by the intertwining property (8.8), the monodromy of $F = (\mathcal{F} \otimes \mathcal{F})[v]$ is equivalent to the braiding of $v \in \mathsf{H}_1 \otimes \overline{\mathsf{H}}_1$. Therefore, it suffices to show that in $\mathsf{H}_1 \otimes \overline{\mathsf{H}}_1$, there exists a unique vector which is invariant under the pure braid group $\mathfrak{PBr}_n$ (up to normalization).

Step 2. By the non-degeneracy of $(\cdot,\cdot)$, we have the following isomorphisms of representations of $\mathfrak{PBr}_n$:

$$\mathsf{H}_1 \otimes \overline{\mathsf{H}}_1 \;\cong\; \mathsf{H}_1 \otimes \mathsf{H}_1^* \;\cong\; \mathrm{Hom}(\mathsf{H}_1,\mathsf{H}_1),$$

where H_1^* is the dual representation. The second isomorphism is canonical, and the first can be found by noticing that the dual and conjugate actions agree on braid generators.

Step 3. Vectors in $\mathsf{H}_1 \otimes \overline{\mathsf{H}}_1$ which are invariant under the pure braid group $\mathfrak{PBr}_n$ correspond with the intertwiners $\mathrm{Hom}_{\mathfrak{PBr}_n}(\mathsf{H}_1,\mathsf{H}_1) \subset \mathrm{Hom}(\mathsf{H}_1,\mathsf{H}_1)$, as in Equation (5.7) in Section 5.7.

Step 4. We prove in [F] that (the group algebra of) the pure braid group $\mathfrak{PBr}_n$ acts irreducibly on H_1. Therefore, by Schur's lemma, we have

$$\dim\big(\mathrm{Hom}_{\mathfrak{PBr}_n}(\mathsf{H}_1,\mathsf{H}_1)\big) = 1.$$

So, up to normalization, there exists exactly one $\mathfrak{PBr}_n$-invariant vector in $\mathsf{H}_1 \otimes \overline{\mathsf{H}}_1$.

Remark. In the case $d_i = 2$ for all i, irreducibility in Step 4 also follows from the quantum Schur-Weyl duality, or by direct calculations. In the general case, the proof in fact is a generalization of the quantum Schur-Weyl duality, and irreducibility of H_1 under the action of $\mathfrak{PBr}_n$ follows as a corollary [F].

Remark. We prove in the articles [D, E] that the dimension of the image of the correspondence map $\mathcal{F}$ equals the cardinality of the set of certain link patterns, example of which is given in Figure 8.6. The uniqueness of the monodromy invariant solution $F(z_1,\ldots,z_n)$ is established in the space $\mathcal{F}[\mathsf{H}_1] \otimes \mathcal{F}[\overline{\mathsf{H}}_1]$. We conjecture that this space is equal to the space of Möbius covariant solutions to the PDE system (8.1) which satisfy an additional power law growth bound, in the spirit of [FK15a]. In the special case when $d_i = 2$ for all i, the PDE system (8.1) (with translation invariance) is equivalent to the second order PDE system (8.5) and the conjecture has been proved in [FK15a, FK15b, B].

Future work related to this thesis

[E] S. M. Flores and E. Peltola. Solution spaces for the Benoit & Saint-Aubin partial differential equations. In preparation.

[F] S. M. Flores and E. Peltola. Higher quantum and classical Schur-Weyl duality for $\mathfrak{sl}_2(\mathbb{C})$. In preparation.

[G] S. M. Flores and E. Peltola. Monodromy invariant CFT correlation functions of first column Kac operators. In preparation.

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